ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\ge\frac{p^2}{4R+r}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}=\sum_{cyc}\frac{r_a^2}{2r+2r_a+\frac{rr_a^3}{p^2r}}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{p^2(4R+r)^2}{6rp^2+2p^2(4R+r)+(4R+r)^3-12Rp^2}\left(\because\sum_{cyc}r_a^3=(4R+r)^3-12Rp^2\right)$$

$$=\frac{p^2(4R+r)^2}{(8R+8r)p^2+(4R+r)^3-12Rp^2}\overset{?}{\ge}\frac{p^2}{4R+r}$$

$$\Leftrightarrow (4R+r)^3\overset{?}{\ge}(8R+8r)p^2+(4R+r)^3-12Rp^2\Leftrightarrow 4R\overset{?}{\ge}8r\to \text{true via Euler}$$

$$\therefore\sum_{cyc}\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\ge\frac{p^2}{4R+r}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$