ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{2r}{h_a^2}+\frac{2}{h_a}+\frac{r}{h_bh_c}}\geq\frac{4p^2r}{p^2+r^2+4Rr}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{2r}{h_a^2}+\frac{2}{h_a}+\frac{r}{h_bh_c}}=\sum_{cyc}\frac{h_a^2}{2r+2h_a+\frac{Rrh_a^3}{2p^2r^2}}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\frac{(p^2+r^2+4Rr)^2}{4R^2}}{6r+\frac{p^2+r^2+4Rr}{R}+\frac{R}{2p^2r}\cdot\sum_{cyc}\frac{a^3b^3}{8R^3}}$$

$$=\frac{4p^2r(p^2+r^2+4Rr)^2}{96R^2r^2p^2+16Rrp^2(p^2+r^2+4Rr)+(p^2+r^2+4Rr)^3-24Rrp^2(p^2+r^2+2Rr)}\overset{?}{\geq}$$

$$\frac{4p^2r}{p^2+r^2+4Rr}\Leftrightarrow(p^2+r^2+4Rr)^3\overset{?}{\geq}96R^2r^2p^2+16Rrp^2(p^2+r^2+4Rr)+$$

$$(p^2+r^2+4Rr)^3-24Rrp^2(p^2+r^2+2Rr)$$

$$\Leftrightarrow3(p^2+r^2+2Rr)\overset{?}{\geq}12Rr+2(p^2+r^2+4Rr)\Leftrightarrow p^2\overset{?}{\geq}14Rr-r^2$$

$$\Leftrightarrow(p^2-16Rr+5r^2)+2r(R-2r)\overset{?}{\geq}0\to\text{true}\because p^2-16Rr+5r^2\overset{\text{Gerretsen}}{\geq}0\text{ and}$$

$$2r(R-2r)\overset{\text{Euler}}{\geq}0\therefore\sum_{cyc}\frac{1}{\frac{2r}{h_a^2}+\frac{2}{h_a}+\frac{r}{h_bh_c}}\geq\frac{4p^2r}{p^2+r^2+4Rr}\ \forall\Delta ABC,$$

" = " iff Δ ABC is equilateral (QED)