ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_a^2+r^2}{r_a^2-r^2}\ge\frac{15}{4}$$

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$$\text{Let } x=\frac{3r}{r_a}, y=\frac{3r}{r_b}, z=\frac{3r}{r_c} \text{ and then : } \sum_{cyc}\frac{r_a^2+r^2}{r_a^2-r^2}\overset{?}{\ge}\frac{15}{4}\Leftrightarrow\sum_{cyc}\frac{\frac{9}{x^2}+1}{\frac{9}{x^2}-1}\underset{(*)}{\overset{?}{\ge}}\frac{15}{4}$$

$$\text{Let } f(t)=\frac{9+t^2}{9-t^2}\ \forall\, t\in(0,3) \text{ and then } f''(t)=\frac{108(t^2+3)}{(9-t^2)^3}>0\Rightarrow f(t)\text{ is convex}$$

$$\text{and so, as } \sum_{cyc}x=\sum_{cyc}\frac{3r}{r_a}=3\Rightarrow 0<x,y,z<3\ \therefore\ \sum_{cyc}\frac{\frac{9}{x^2}+1}{\frac{9}{x^2}-1}\overset{\text{Jensen}}{\ge}3.\frac{9+\left(\frac{\sum_{cyc}x}{3}\right)^2}{9-\left(\frac{\sum_{cyc}x}{3}\right)^2}$$

$$=3.\frac{10}{8}=\frac{15}{4}\Rightarrow(*)\text{ is true }\therefore\ \sum_{cyc}\frac{r_a^2+r^2}{r_a^2-r^2}\ge\frac{15}{4}\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$