ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$3 \le \sum_{cyc} \frac{m_a^2}{r_b r_c} \le \frac{R^2 + 2r^2}{Rr}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{m_a^2}{r_b r_c} = \sum_{cyc} \frac{4m_a^2(-s^2 + sa + bc)}{4r^2s^2}$$

$$= \frac{1}{4r^2s^2}\left(\begin{array}{c} -6s^2(s^2 - 4Rr - r^2) + s\sum_{cyc}\left(a\left(2\sum_{cyc} a^2 - 3a^2\right)\right) + \\ \sum_{cyc}\left(bc\left(2\sum_{cyc} a^2 - 3a^2\right)\right) \end{array}\right)$$

$$= \frac{\begin{array}{c} -6s^2(s^2 - 4Rr - r^2) + 8s^2(s^2 - 4Rr - r^2) - 6s^2(s^2 - 6Rr - 3r^2) + \\ 4(s^4 - (4Rr + r)^2) - 24Rrs^2 \end{array}}{4r^2s^2}$$

$$= \frac{(R + 4r)s^2 - r(4R + r)^2}{rs^2} \overset{?}{\le} \frac{R^2 + 2r^2}{Rr} \Leftrightarrow R(4R + r)^2 \overset{?}{\ge} (4R - 2r)s^2$$

$$\to \text{true via Blundon} - \text{Gerretsen} \therefore \sum_{cyc} \frac{m_a^2}{r_b r_c} \le \frac{R^2 + 2r^2}{Rr} \text{ and again,}$$

$$\sum_{cyc} \frac{m_a^2}{r_b r_c} \overset{\text{Lascu + AM-GM}}{\ge} \sum_{cyc} \frac{s(s-a)(s-b)(s-c)}{r^2s^2} = \frac{3r^2s^2}{r^2s^2} = 3 \text{ and so,}$$

$$3 \le \sum_{cyc} \frac{m_a^2}{r_b r_c} \le \frac{R^2 + 2r^2}{Rr}\ \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$