ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$3 \leq \sum_{cyc} \frac{m_a^2}{h_b h_c} \leq 3\left(\frac{R}{2r}\right)^3$$

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$$\sum_{cyc} \frac{m_a^2}{h_b h_c} = \frac{1}{16r^2s^2} \cdot \sum_{cyc}\left(bc\left(2\sum_{cyc} a^2 - 3a^2\right)\right)$$

$$= \frac{4(s^4 - (4Rr + r)^2) - 3(4Rrs)(2s)}{16r^2s^2} \overset{?}{\leq} 3\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow (3R^3 + 12Rr^2)s^2 + 2r^3(4R + r)^2 \underset{(*)}{\overset{?}{\geq}} 2rs^4$$

$$\text{Now, } 2rs^4 \overset{\text{Gerretsen}}{\leq} 2rs^2(4R^2 + 4Rr + 3r^2) \overset{?}{\leq} \text{LHS of } (*) \Leftrightarrow$$

$$(3R^3 - 8R^2r + 4Rr^2 - 6r^3)s^2 + 2r^3(4R + r)^2 \underset{(**)}{\overset{?}{\geq}} 0 \text{ and it's trivially true when :}$$

$$3R^3 - 8R^2r + 4Rr^2 - 6r^3 \geq 0 \text{ and when : } 3R^3 - 8R^2r + 4Rr^2 - 6r^3 < 0,$$

$$\text{LHS of } (**) \overset{\text{Gerretsen}}{\geq} (3R^3 - 8R^2r + 4Rr^2 - 6r^3)(4R^2 + 4Rr + 3r^2) +$$

$$2r^3(4R + r)^2 \overset{?}{\geq} 0 \Leftrightarrow 12t^5 - 20t^4 - 7t^3 + 4t - 16 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)(12t^4 + 4t^3 + t^2 + 2t + 8) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true } \forall \Delta ABC \therefore \sum_{cyc} \frac{m_a^2}{h_b h_c} \leq 3\left(\frac{R}{2r}\right)^3 \text{ and again, } \sum_{cyc} \frac{m_a^2}{h_b h_c} \geq \sum_{cyc} \frac{m_a^2}{w_b w_c}$$

$$\overset{\text{Lascu + AM-GM}}{\geq} \sum_{cyc} \frac{s(s-a)}{\sqrt{s(s-b)} \cdot \sqrt{s(s-c)}} \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\frac{s(s-a).s(s-b).s(s-c)}{s(s-a).s(s-b).s(s-c)}} = 3$$

$$\text{and so, } 3 \leq \sum_{cyc} \frac{m_a^2}{h_b h_c} \leq 3\left(\frac{R}{2r}\right)^3 \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$