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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_a^2-r^2}\geq\frac{3}{8r^2}$$

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$$\text{Let } x=\frac{3r}{h_a}, y=\frac{3r}{h_b}, z=\frac{3r}{h_c} \text{ and then : } \sum_{cyc}\frac{1}{h_a^2-r^2}\overset{?}{\geq}\frac{3}{8r^2}\Leftrightarrow\sum_{cyc}\frac{1}{\frac{9}{x^2}-1}\underset{(*)}{\overset{?}{\geq}}\frac{3}{8}$$

$$\text{Let } f(t)=\frac{t^2}{9-t^2}\ \forall\, t\in(0,3) \text{ and then } f''(t)=\frac{54(t^2+3)}{(9-t^2)^3}>0 \text{ and so,}$$

$$\text{as }\sum_{cyc}x=\sum_{cyc}\frac{3r}{h_a}=3\Rightarrow x,y,z<3\ \therefore\ \sum_{cyc}\frac{1}{\frac{9}{x^2}-1}\overset{\text{Jensen}}{\geq}\frac{3}{\frac{9}{\left(\frac{\sum_{cyc}x}{3}\right)^2}-1}=\frac{3}{9-1}=\frac{3}{8}$$

$$\Rightarrow (*) \text{ is true }\therefore\ \sum_{cyc}\frac{1}{h_a^2-r^2}\geq\frac{3}{8r^2}\ \forall\,\Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$