ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$18\sum_{cyc}\frac{r^3}{h_a^3}\ge 1+3\sum_{cyc}\frac{r^2}{h_a^2}$$

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$$\text{Let } x=\frac{3r}{h_a}, y=\frac{3r}{h_b}, z=\frac{3r}{h_c} \text{ and then : } 18\sum_{cyc}\frac{r^3}{h_a^3}\overset{?}{\ge} 1+3\sum_{cyc}\frac{r^2}{h_a^2}$$

$$\Leftrightarrow \frac{18}{27}.\sum_{cyc}x^3\overset{?}{\ge} 1+\frac{3}{9}.\sum_{cyc}x^2$$

$$\Leftrightarrow 18\sum_{cyc}x^3\overset{?}{\ge}\left(\sum_{cyc}x\right)^3+3\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\left(\because \sum_{cyc}x=\sum_{cyc}\frac{3r}{h_a}=3\right)$$

$$\to \text{true} \because 9\sum_{cyc}x^3\overset{\text{Holder}}{\ge}\left(\sum_{cyc}x\right)^3 \text{ and } 3\sum_{cyc}x^3\overset{\text{Chebyshev}}{\ge}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\therefore 18\sum_{cyc}\frac{r^3}{h_a^3}\ge 1+3\sum_{cyc}\frac{r^2}{h_a^2}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$