ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{3}.\left(\frac{2r}{R}\right) \le \sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \le \sqrt{3}\left(\frac{R}{2r}\right)^2$$

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$$\sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} = 2R.\sum_{cyc} \frac{(a+b)^2}{(a+b)(b+c)(c+a)}$$

$$= \frac{4R}{2s(s^2+2Rr+r^2)}.\left(\sum_{cyc} a^2 + \sum_{cyc} ab\right)$$

$$\therefore \sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} = \frac{2R(3s^2-4Rr-r^2)}{s(s^2+2Rr+r^2)} \to \textcircled{1} \text{ and}$$

$$\because s \overset{\text{Mitrinovic}}{\ge} 3\sqrt{3}r \therefore \text{in order to prove :}$$

$$\sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \overset{?}{\le} \sqrt{3}.\left(\frac{R}{2r}\right)^2, \text{it suffices to prove :}$$

$$9R(s^2+2Rr+r^2) \overset{?}{\ge} 8r(3s^2-4Rr-r^2)$$

$$\Leftrightarrow (9R-24r)s^2 + r(18R^2+41Rr+8r^2) \underset{(*)}{\overset{?}{\ge}} 0 \text{ and it's trivially true when :}$$

$$9R-24r \ge 0 \text{ and when : } 9R-24r < 0, \text{LHS of } (*) \overset{\text{Gerretsen}}{\ge}$$

$$(9R-24r)(4R^2+4Rr+3r^2) + r(18R^2+41Rr+8r^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow 18t^3 - 21t^2 - 14t - 32 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(18t^2+15t+16) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true } \forall \Delta \text{ ABC } \therefore \sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \le \sqrt{3}.\left(\frac{R}{2r}\right)^2$$

$$\text{Again, since } s \overset{\text{Mitrinovic}}{\le} \frac{3\sqrt{3}R}{2} \therefore \textcircled{1} \Rightarrow \text{in order to prove :}$$

$$\sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \overset{?}{\ge} \sqrt{3}.\left(\frac{2r}{R}\right), \text{it suffices to prove :}$$

$$2R(3s^2-4Rr-r^2) \overset{?}{\ge} 9r(s^2+2Rr+r^2)$$

$$\Leftrightarrow (6R-9r)s^2 - r(8R^2+20Rr+9r^2) \underset{(**)}{\overset{?}{\ge}} 0 \text{ and once again, LHS of } (**) \overset{\text{Gerretsen}}{\ge}$$

$$(6R-9r)(16Rr-5r^2) - r(8R^2+20Rr+9r^2) = 2r(R-2r)(44R-9r) \ge 0$$

$$\because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \text{ is true } \therefore \sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \ge \sqrt{3}.\left(\frac{2r}{R}\right)$$

$$\textbf{and so}, \sqrt{3}.\left(\frac{2r}{R}\right) \leq \sum_{cyc} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \leq \sqrt{3}.\left(\frac{R}{2r}\right)^2 \quad \forall \Delta\, ABC,$$

$'' ='' $ **iff** Δ **ABC is equilateral (QED)**