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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{bc(h_a-r)}\ge\sum_{cyc}\frac{1}{bc(r_a-r)}$$

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$$\sum_{cyc}\frac{1}{bc(h_a-r)}=\sum_{cyc}\frac{1}{bc\left(\frac{2rs}{a}-r\right)}=\sum_{cyc}\frac{a^2}{rabc(b+c)}=$$

$$=\frac{1}{4Rr^2s}\cdot\frac{1}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(a^2\left(a^2+\sum_{cyc}ab\right)\right)$$

$$=\frac{1}{8Rr^2s^2(s^2+2Rr+r^2)}\cdot\left(2\sum_{cyc}a^2b^2-16r^2s^2+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\right)$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2+s^4-(4Rr+r^2)^2}{4Rr^2s^2(s^2+2Rr+r^2)}\overset{?}{\ge}\sum_{cyc}\frac{1}{bc(r_a-r)}$$

$$=\sum_{cyc}\frac{1}{bc\left(\frac{rs}{s-a}-\frac{rs}{s}\right)}=\sum_{cyc}\frac{s-a}{r.4Rrs}=\frac{1}{4Rr^2}\Leftrightarrow s^2(s^2-10Rr-7r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow s^2\left(s^2-16Rr+5r^2+6r(R-2r)\right)\overset{?}{\ge}0\to\text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0$$

$$\text{and } R\overset{\text{Euler}}{\ge}2r\therefore\sum_{cyc}\frac{1}{bc(h_a-r)}\ge\sum_{cyc}\frac{1}{bc(r_a-r)}\ \forall\,\Delta\text{ ABC},$$

$''=''$ iff Δ ABC is equilateral (QED)