ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{2R^2r} \le \sum_{cyc} \frac{1}{bc(h_a - r)} \le \frac{R}{16r^4}$$

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$$\sum_{cyc} \frac{1}{bc(h_a - r)} = \sum_{cyc} \frac{1}{bc\left(\frac{2rs}{a} - r\right)} = \sum_{cyc} \frac{a^2}{rabc(b+c)}$$

$$= \frac{1}{4Rr^2s} \cdot \frac{1}{2s(s^2 + 2Rr + r^2)} \cdot \sum_{cyc}\left(a^2\left(a^2 + \sum_{cyc} ab\right)\right)$$

$$= \frac{1}{8Rr^2s^2(s^2 + 2Rr + r^2)} \cdot \left(2\sum_{cyc} a^2b^2 - 16r^2s^2 + \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)\right)$$

$$= \frac{(s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 8r^2s^2 + s^4 - (4Rr + r^2)^2}{4Rr^2s^2(s^2 + 2Rr + r^2)} \overset{?}{\le} \frac{R}{16r^4}$$

$$\Leftrightarrow (R^2 - 8r^2)s^2 + r(2R^3 + R^2r + 32Rr^2 + 24r^3) \underset{(*)}{\overset{?}{\ge}} 0 \text{ and it's trivially true when :}$$

$$R^2 - 8r^2 \ge 0 \text{ and when : } R^2 - 8r^2 < 0, \text{LHS of } (*) \overset{\text{Gerretsen}}{\ge}$$

$$(R^2 - 8r^2)(4R^2 + 4Rr + 3r^2) + r(2R^3 + R^2r + 32Rr^2 + 24r^3) \overset{?}{\ge} 0$$

$$\Leftrightarrow 2R^2(R - 2r)(2R + 7r) \overset{?}{\ge} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{1}{bc(h_a - r)} \le \frac{R}{16r^4} \text{ and again, } \sum_{cyc} \frac{1}{bc(h_a - r)} \overset{?}{\ge} \frac{1}{2R^2r} \Leftrightarrow$$

$$\frac{(s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 8r^2s^2 + s^4 - (4Rr + r^2)^2}{4Rr^2s^2(s^2 + 2Rr + r^2)} \overset{?}{\ge} \frac{1}{2R^2r}$$

$$\Leftrightarrow (R - r)s^2 \underset{(**)}{\overset{?}{\ge}} r(4R^2 + 5Rr + r^2)$$

$$\text{Indeed, } (R - r)s^2 \overset{\text{Gerretsen}}{\ge} (R - r)(16Rr - 5r^2) \overset{?}{\ge} r(4R^2 + 5Rr + r^2)$$

$$\Leftrightarrow 2r(R - 2r)(6R - r) \overset{?}{\ge} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \text{ is true}$$

$$\therefore \frac{1}{2R^2r} \le \sum_{cyc} \frac{1}{bc(h_a - r)} \le \frac{R}{16r^4} \ \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$