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In $\Delta ABC$ the following relationship holds:

$$\sum_{cyc} h_a \sin^2 A \le \sum_{cyc} r_a \sin^2 A$$

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$$In\ \Delta ABC\ \ wlog\ \ a \le b \le c\ ,\qquad h_a \ge h_b \ge h_c$$

$$and\ \ r_a \le r_b \le r_c$$

$$LHS = \sum_{cyc} h_a \sin^2 A \overset{Chebyshev}{\overbrace{\le}} \frac{1}{3}\sum_{cyc} h_a \cdot \sum_{cyc} \sin^2 A \quad (*)$$

$$RHS = \sum_{cyc} r_a \sin^2 A \overset{Chebyshev}{\overbrace{\ge}} \frac{1}{3}\sum_{cyc} r_a \cdot \sum_{cyc} \sin^2 A \quad (**)$$

$$Let's\ prove\ \ that:\ \sum_{cyc} h_a \le \sum_{cyc} r_a$$

$$For\ \ this\ :$$

$$\sum_{cyc} h_a = \frac{S^2 + r^2 + 4Rr}{2R} \overset{Gerretsen}{\overbrace{\le}} \frac{4R^2 + 4Rr + 3r^2 + r^2 + 4Rr}{2R} =$$

$$= \frac{4R^2 + 8Rr + 4r^2}{2R} = \frac{2(r+R)^2}{R}\ ;\ \sum_{cyc} r_a = 4R + r$$

$$\frac{2(r+R)^2}{R} \overset{?}{\overbrace{\le}} 4R + r \to \ 2R^2 + 4Rr + 2r^2 \overset{?}{\overbrace{\le}} 4R + r\ \to$$

$$2R^2 - 3Rr - 2r^2 \overset{?}{\overbrace{\ge}} 0\ \ \to\ \ (R - 2r)(2R + r) \ge 0 \to\ \ R \ge 2r\ (EULER)$$

$$Equality\ \ holds\ for\ :\ \ a = b = c$$