ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{13}{4}-\frac{r}{2R}\le\sum\frac{r_br_c}{w_a^2}\le\frac{3R}{2r}$$

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Solution by Tapas Das-India

$$w_a^2=\frac{4bcs(s-a)}{(b+c)^2}$$

$$\frac{r_br_c}{w_a^2}=\frac{\frac{F^2}{(s-b)(s-c)}}{\frac{4bcs(s-a)}{(b+c)^2}}=\frac{F^2}{s(s-a)(s-b)(s-c)}\cdot\frac{(b+c)^2}{4bc}=\frac{(b+c)^2}{4bc}$$

$$\frac{r_br_c}{w_a^2}=\frac{(b+c)^2}{4bc}=\frac{a(b+c)^2}{4abc}=\frac{a(a^2+b^2+c^2)+2abc-a^3}{4abc}=$$
$$=\frac{1}{4}\left(\frac{a(a^2+b^2+c^2)-a^3}{abc}+2\right)$$

$$\sum\frac{r_br_c}{w_a^2}=\frac{1}{4}\left(\frac{(\sum a.\sum a^2-\sum a^3)}{abc}+6\right)=$$
$$=\frac{1}{4}\left(\frac{2s.2(s^2-r^2-4Rr)-2s(s^2-3r^2-6Rr)}{4Rrs}+6\right)=$$
$$=\frac{1}{4}\left(\frac{s^2+r^2+10Rr}{2Rr}\right)\overset{Gerretsn}{\ge}\frac{1}{4}\left(\frac{16Rr-5r^2+r^2+10Rr}{2Rr}\right)=$$
$$=\frac{(26Rr-4r^2)}{8Rr}=\frac{13}{4}-\frac{r}{2R}$$

$$\sum\frac{r_br_c}{w_a^2}=\frac{1}{4}\sum\frac{(b+c)^2}{bc}=\frac{1}{4}\sum\left(\frac{b}{c}+\frac{c}{b}+2\right)\overset{Bandila\,\&\,Euler}{\le}\frac{1}{4}\sum\left(\frac{R}{r}+\frac{R}{r}\right)=\frac{3R}{2r}$$

Equality holds for an equilateral triangle.