ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a}{b+c}+\lambda\prod_{cyc}\frac{w_a}{r_a}\geq\lambda+\frac{3}{2},\lambda\leq\frac{1}{2}$$

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$$w_a\leq\sqrt{r_b r_b}\ \text{and analogs}\Rightarrow\prod_{cyc}\frac{w_a}{r_a}\leq 1\therefore\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}-\lambda\left(1-\prod_{cyc}\frac{w_a}{r_a}\right)$$

$$\overset{\lambda\leq\frac{1}{2}}{\geq}\sum_{cyc}\frac{a(a^2+\sum_{cyc}ab)}{2s(s^2+2Rr+r^2)}-\frac{3}{2}-\frac{1}{2}\left(1-\frac{16Rr^2s^2}{(s^2+2Rr+r^2)rs^2}\right)$$

$$=\frac{2s(s^2-6Rr-3r^2)+2s(s^2+4Rr+r^2)-3s(s^2+2Rr+r^2)}{2s(s^2+2Rr+r^2)}-$$

$$\frac{1}{2}\cdot\frac{s^2-14Rr+r^2}{s^2+2Rr+r^2}=\frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)}-\frac{s^2-14Rr+r^2}{2(s^2+2Rr+r^2)}=\frac{4r(R-2r)}{2(s^2+2Rr+r^2)}$$

$$\overset{\text{Euler}}{\geq}0\therefore\sum_{cyc}\frac{a}{b+c}+\lambda\prod_{cyc}\frac{w_a}{r_a}\geq\lambda+\frac{3}{2}\ \forall\ \Delta\ \text{ABC and}\ \forall\ \lambda\leq\frac{1}{2},$$

$$''=''\ \text{iff}\ \Delta\ \text{ABC is equilateral (QED)}$$