ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{AH}{ah_a} \le \sum \frac{AH}{ar_a}$$

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$$Known\ results: 1) AH = 2R\cos A, 2) r_a = s\tan\frac{A}{2}$$

$$3) \sum \tan^2\frac{A}{2}.\tan^2\frac{B}{2} = \frac{s^2 - 2r^2 - 8Rr}{s^2} \quad 4) \prod \tan^2\frac{A}{2} = \left(\frac{r}{s}\right)^2$$

$$\sum \cot^2\frac{A}{2} = \frac{\sum \tan^2\frac{A}{2}.\tan^2\frac{B}{2}}{\prod \tan^2\frac{A}{2}} = \frac{s^2 - 2r^2 - 8Rr}{r^2}$$

$$\frac{AH}{ar_a} = \frac{2R\cos A}{2RsinA.s\tan\frac{A}{2}} = \frac{cotA}{s\tan\frac{A}{2}} = \frac{\frac{\cot^2\frac{A}{2}-1}{2\cot\frac{A}{2}}}{.s\tan\frac{A}{2}} = \frac{1}{2s}\left(\cot^2\frac{A}{2} - 1\right)$$

$$\sum \frac{AH}{ar_a} = \frac{1}{2s}\sum\left(\cot^2\frac{A}{2} - 1\right) = \frac{1}{2s}\left(\frac{s^2 - 2r^2 - 8Rr}{r^2} - 3\right) =$$

$$= \frac{1}{2s}\frac{s^2 - 5r^2 - 8Rr}{r^2} \overset{Gerrestn}{\ge} \frac{1}{2s}\frac{16Rr - 5r^2 - 5r^2 - 8Rr}{r^2} = \frac{1}{s}\left(\frac{4R}{r} - 5\right)$$

$$\sum \frac{AH}{ah_a} = \sum \frac{2R\,cosA}{a.\frac{2F}{a}} = \frac{R}{F}\sum cosA = \frac{R}{r.s}\left(1 + \frac{r}{R}\right) \overset{Euler}{\le} \frac{3R}{2rs}$$

$$We\ need\ to\ show:$$

$$\frac{1}{s}\left(\frac{4R}{r} - 5\right) \ge \frac{3R}{2rs}\ \ or, \frac{4R}{r} - \frac{3R}{2r} \ge 5\ or, \frac{5R}{2r} \ge 5\ or, R \ge 2r\ true\ by\ Euler.$$

Equality holds for an equilateral triangle.