# ROMANIAN MATHEMATICAL MAGAZINE

**In $\Delta ABC$ the following relationship holds:**

$$\frac{1}{s}\left(\frac{4R}{r}-5\right)\le\sum\frac{AH}{ar_a}\le\frac{1}{s}\left(\frac{2R^2}{r^2}-5\right)$$

***Proposed by Marin Chirciu-Romania***

***Solution by Tapas Das-India***

$$Known\ Results: 1)AH=2R\cos A,2)r_a=s\tan\frac{A}{2}$$

$$3)\sum\tan^2\frac{A}{2}.\tan^2\frac{B}{2}=\frac{s^2-2r^2-8Rr}{s^2}\quad 4)\prod\tan^2\frac{A}{2}=\left(\frac{r}{s}\right)^2$$

$$\sum\cot^2\frac{A}{2}=\frac{\sum\tan^2\frac{A}{2}.\tan^2\frac{B}{2}}{\prod\tan^2\frac{A}{2}}=\frac{s^2-2r^2-8Rr}{r^2}$$

$$\frac{AH}{ar_a}=\frac{2R\cos A}{2RsinA.s\tan\frac{A}{2}}=\frac{cotA}{s\tan\frac{A}{2}}=\frac{\frac{\cot^2\frac{A}{2}-1}{2\cot\frac{A}{2}}}{s\tan\frac{A}{2}}=\frac{1}{2s}\left(\cot^2\frac{A}{2}-1\right)$$

$$\sum\frac{AH}{ar_a}=\frac{1}{2s}\sum\left(\cot^2\frac{A}{2}-1\right)=\frac{1}{2s}\left(\frac{s^2-2r^2-8Rr}{r^2}-3\right)=$$

$$=\frac{1}{2s}\frac{s^2-5r^2-8Rr}{r^2}\overset{Gerretsen}{\le}\frac{1}{2s}\frac{4R^2+4Rr+3r^2-5r^2-8Rr}{r^2}=$$

$$=\frac{1}{2s}\left(\frac{4R^2}{r^2}-\frac{4R}{r}-2\right)\overset{Euler}{\le}\frac{1}{2s}\left(\frac{4R^2}{r^2}-4\times2-2\right)=\frac{1}{s}\left(\frac{2R^2}{r^2}-5\right)$$

$$\sum\frac{AH}{ar_a}=\frac{1}{2s}\frac{s^2-5r^2-8Rr}{r^2}\overset{Gerretsen}{\ge}\frac{1}{2s}\frac{16Rr-5r^2-5r^2-8Rr}{r^2}=\frac{1}{s}\left(\frac{4R}{r}-5\right)$$

**Equality holds for an equilateral triangle.**