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In ΔABC the following relationship holds:

$$\left(\sum_{cyc} rr_a\right)\left(\sum_{cyc}\frac{a^2}{h_a}\right) \le 9\frac{R^3(2R-r)}{2r}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^2}{h_a} = \sum_{cyc}\frac{a^2}{\frac{2F}{a}} = \sum_{cyc}\frac{a^3}{2F} = \frac{2(S^3-3Sr^2-6SRr)}{2Sr} =$$

$$\frac{S^2-3r^2-6Rr}{2r} \overset{Gerretsen}{\le} \frac{4R^2+4Rr+3r^2-3r^2-6Rr}{r} =$$

$$= \frac{4R^2-2Rr}{r} = \frac{2R(2R-r)}{r} \quad (*)$$

$$\sum_{cyc} rr_a = r(4R+r) \overset{Euler}{\le} \frac{9Rr}{2} \quad (**)$$

From $(*)$ *and* $(**)$ *we have*

$$\left(\sum_{cyc} rr_a\right)\left(\sum_{cyc}\frac{a^2}{h_a}\right) \overset{(*),(**)}{\le} \frac{9Rr}{2}\cdot\frac{2R(2R-r)}{r} \overset{Euler}{\le} 9\frac{R^3(2R-r)}{2r}$$

Equality holds for $a=b=c$