ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{3r}{2R}\le\sum\frac{a}{2a+b+c}\le\frac{3}{4}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\sum\frac{a}{2a+b+c}=\sum\frac{a}{(a+b)+(a+c)}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{a}{a+c}\right)=$$

$$=\frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{b}{a+b}\right)=\frac{1}{4}\sum\frac{a+b}{a+b}=\frac{3}{4}$$

$$abc=4Rrs\overset{Euler\ \&\ Mitrinovic}{\ge}4.2r.r.3\sqrt{3}r=\left(2\sqrt{3}r\right)^3(1)$$

$$\sum\frac{a}{2a+b+c}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{abc}{(2a+b+c)(2b+c+a)(2c+a+b)}}\overset{AM-GM}{\ge}$$

$$\ge\frac{\sqrt[3]{abc}}{\frac{2a+b+c+2b+c+a+2c+a+b}{3}}=\frac{3\sqrt[3]{abc}}{4(a+b+c)}\overset{(1)}{\ge}3\times\frac{2\times3\sqrt{3}r}{8s}\overset{Mitrinovic}{\ge}$$

$$\ge 3\times\frac{2\times3\sqrt{3}r}{8\times\frac{3\sqrt{3}R}{2}}=\frac{3r}{2R}$$

Equality holds for an equilateral triangle.