ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) \le 9\frac{R^2}{r}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

First, let us prove that:

$$\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)} \le \frac{2R}{r}$$

$$For\ this\ :\ \sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)} = 1+\frac{(4R+r)^2}{S^2} \overset{Gerretsen}{\le}$$

$$1+\frac{(4R+r)^2}{16Rr-r^2} = \frac{16R^2+24Rr-4r^2}{16Rr-5r^2} \overset{?}{\le} \frac{2R}{r}$$

$$8R^2+12Rr-2r^2 \le 16Rr-5r^2$$

$$8R^2-17Rr+2r^2 \ge 0$$

$$(R-2r)(8R-r) \ge 0\ \to\ R \ge 2r\ (Euler)\ True$$

$$Now:\ \left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) \le (4R+r).\frac{2R}{r} \overset{Euler}{\le} \frac{9R}{2}.\frac{2R}{r} = \frac{9R^2}{r}$$

Equality holds for an equilateral triangle.