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In acute ΔABC the following relationship holds:

$$\cos(A).\cos(B)+\cos(C).\cos(B)+\cos(A).\cos(C)\le\frac{a^3+b^3+c^3}{16RS}$$

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$$RHS=\frac{a^3+b^3+c^3}{16RS}=\frac{a^3+b^3+c^3}{4abc}\overset{AM-GM}{\overbrace{\ge}}\frac{3abc}{4abc}=\frac{3}{4}\quad(*)$$

$$LHS=\sum_{cyc}\cos(A).\cos(B)=\frac{p^2+r^2-4R^2}{4R^2}\overset{Gerretsen}{\overbrace{\le}}$$

$$\frac{4R^2+4Rr+3r^2+r^2-4R^2}{4R^2}=\frac{4Rr+4r^2}{4R^2}=\frac{r}{R}+\left(\frac{r}{R}\right)^2\le\frac{1}{2}+\frac{1}{4}=\frac{3}{4}\quad(**)$$

$$LHS\overset{(**)}{\overbrace{\le}}\frac{3}{4},\quad RHS\overset{(*)}{\overbrace{\ge}}\frac{3}{4}\ \rightarrow\ LHS\le RHS\ (PROVED)$$

Equality holds for an equilateral triangle.