ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a^4 h_a}{m_a + h_a m_a^2} + \frac{r_b^4 h_b}{m_b + h_b m_b^2} + \frac{r_c^4 h_c}{m_c + h_c m_c^2} \geq \frac{1944 r^5}{4R + 9R^3}$$

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Solution by Tapas Das-India

$$\sum \frac{m_a}{h_a} \overset{Chebyshev}{\leq} \frac{1}{3}\left(\sum m_a\right)\left(\sum \frac{1}{h_a}\right) \overset{Gotman}{\leq} \frac{1}{3} \cdot \frac{9R}{2} \cdot \frac{1}{r} = \frac{3R}{2r},$$

$$\sum m_a^2 = \frac{3}{4}\left(\sum a^2\right) \overset{Leibniz}{\leq} \frac{27R^2}{4}$$

$$\sum r_a^2 = \left(\sum r_a\right)^2 - 2\sum r_a r_b = (4R + r)^2 - 2s^2 \overset{Doucet}{\geq}$$

$$\geq 3s^2 - 2s^2 = s^2 \overset{Mitrinovic}{\geq} 27r^2$$

$$\frac{r_a^4 h_a}{m_a + h_a m_a^2} + \frac{r_b^4 h_b}{m_b + h_b m_b^2} + \frac{r_c^4 h_c}{m_c + h_c m_c^2} = \sum \frac{r_a^4 h_a}{m_a + h_a m_a^2} =$$

$$= \sum \frac{r_a^4}{\frac{m_a}{h_a} + m_a^2} \overset{Bergstrom}{\geq} \frac{\left(\sum r_a^2\right)^2}{\sum \frac{m_a}{h_a} + \sum m_a^2} \geq \frac{(27r^2)^2}{\frac{3R}{2r} + \frac{27R^2}{4}} = \frac{243r^5 \times 4}{2R + 9R^2 r} \overset{Euler}{\geq}$$

$$\geq \frac{243r^5 \times 4}{2R + \frac{9R^2 R}{2}} = \frac{1944r^5}{4R + 9R^3}$$

Equality holds for an equilateral triangle.