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If I_a, I_b, I_c − excenters in ΔABC then prove that:

$$\frac{2}{F}\left(2-\frac{r}{R}\right)\le\sum_{cyc}\frac{1}{F_{BCI_a}}\le\frac{2}{F}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Proposed by Eldeniz Hesenov-Georgia

Solution by Tapas Das-India

$$F_{BCI_a}=\frac{1}{2}\cdot r_a\cdot a=\frac{F}{2}\frac{a}{s-a}$$

$$\sum_{cyc}\frac{1}{F_{BCI_a}}=\frac{2}{F}\sum\frac{s-a}{a}=\frac{2}{F}\left(s\sum\frac{1}{a}-3\right)=\frac{2}{F}\left(s\frac{ab+bc+ca}{abc}-3\right)=$$

$$=\frac{2}{F}\left(\frac{s^2+r^2+4Rr}{4Rr}-3\right)=\frac{2}{F}\left(\frac{s^2+r^2-8Rr}{4Rr}\right)\overset{Gerretsen}{\ge}$$

$$\ge\frac{2}{F}\left(\frac{16Rr-5r^2+r^2-8Rr}{4Rr}\right)=\frac{2}{F}\left(\frac{8Rr-4r^2}{4Rr}\right)=\frac{2}{F}\left(2-\frac{r}{R}\right)$$

$$\sum_{cyc}\frac{1}{F_{BCI_a}}=\frac{2}{F}\left(\frac{s^2+r^2-8Rr}{4Rr}\right)\overset{Gerretsen}{\le}\frac{2}{F}\left(\frac{4R^2+4Rr+3r^2+r^2-8Rr}{4Rr}\right)=$$

$$=\frac{2}{F}\left(\frac{4R^2-4Rr+4r^2}{4Rr}\right)=\frac{2}{F}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Equality holds for an equilateral triangle.