ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\ge (3r)^{m_a+m_b+m_c}$$

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Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x\ln x$

$$f'(x)=\ln x+x\cdot\frac{1}{x}=\ln x+1; f''(x)=\frac{1}{x}>0\Rightarrow f \text{ convex}$$

By Jensen's inequality:

$$f(m_a)+f(m_b)+f(m_c)\ge 3f\left(\frac{m_a+m_b+m_c}{3}\right)$$

$$m_a\ln m_a+m_b\ln m_b+m_c\ln m_c\ge 3\,\frac{m_a+m_b+m_c}{3}\cdot\ln\left(\frac{m_a+m_b+m_c}{3}\right)$$

$$\ln\left(m_a^{m_a}\right)+\ln\left(m_b^{m_b}\right)+\ln\left(m_c^{m_c}\right)\ge (m_a+m_b+m_c)\ln\left(\frac{m_a+m_b+m_c}{3}\right)\quad (1)$$

$$h_a\le m_a; h_b\le m_b; h_c\le m_c\Rightarrow$$

$$\frac{1}{h_a}\ge\frac{1}{m_a};\frac{1}{h_b}\ge\frac{1}{m_c};\frac{1}{h_c}\ge\frac{1}{m_c}$$

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\ge\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}$$

$$\frac{1}{r}\ge\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\overset{BERGSTROM}{\ge}\frac{(1+1+1)^2}{m_a+m_b+m_c}=$$

$$=\frac{9}{m_a+m_b+m_c}\Rightarrow m_a+m_b+m_c\ge 9r\Rightarrow\frac{m_a+m_b+m_c}{3}\ge 3r\quad (2)$$

$$\ln\left(\frac{m_a+m_b+m_c}{3}\right)\ge\ln(3r)$$

By (1):

$$\ln\left(m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\right)\ge (m_a+m_b+m_c)\ln\left(\frac{m_a+m_b+m_c}{3}\right)\ge$$

$$\overset{(2)}{\ge}(m_a+m_b+m_c)\ln(3r)=\ln(3r)^{m_a+m_b+m_c}$$

$$\ln\left(m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\right)\ge\ln(3r)^{m_a+m_b+m_c}$$

$$m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\ge(3r)^{m_a+m_b+m_c}$$

Equality holds for: $a=b=c$.