ROMANIAN MATHEMATICAL MAGAZINE

$$If\ in\ \Delta ABC\ , A' \in (BC), B' \in (CA), C' \in (AB)\ and$$

$$\frac{BA'}{CA'} = \frac{CB'}{AB'} = \frac{AC'}{BC'} = 5\ \ then\ prove\ that:$$

$$AA'^2 + BB'^2 + CC'^2 > \frac{31\sqrt{3}}{9}F$$

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Solution by Tapas Das-India

$$BC = a, CA = b, AB = c\ , \frac{BA'}{CA'} = \frac{CB'}{AB'} = \frac{AC'}{BC'} = 5$$

$$then\ BA' : CA' = 5:1 \Rightarrow\ BA' = \frac{5a}{6}, CA' = \frac{a}{6}$$

$$from\ \Delta ABA'\ we\ have\ AA'^2 = AB^2 + BA'^2 - 2AB.BA' \cos\angle ABA'$$

$$AA'^2 = c^2 + \left(\frac{5a}{6}\right)^2 - 2c.\frac{5a}{6}\cos B \overset{cosB<1}{>} c^2 + \frac{25a^2}{36} - \frac{10ac}{6}$$

$$Similary:$$

$$BB'^2 = a^2 + \frac{25b^2}{36} - \frac{10ab}{6}, CC' = b^2 + \frac{25c^2}{36} - \frac{10bc}{6}$$

$$AA'^2 + BB'^2 + CC'^2 > c^2 + \frac{25a^2}{36} - \frac{10ac}{6} + a^2 + \frac{25b^2}{36} - \frac{10ab}{6} + b^2 + \frac{25c^2}{36} - \frac{10bc}{6} =$$

$$= \frac{61}{36}(a^2 + b^2 + c^2) - \frac{60}{36}(ab + bc + ca) > \frac{61}{36}(a^2 + b^2 + c^2) - \frac{60}{36}(a^2 + b^2 + c^2)$$

$$= \frac{31}{36}(a^2 + b^2 + c^2) > \frac{31}{36}(ab + bc + ca) \overset{Gordon}{\geq} \frac{31}{36}.4\sqrt{3}F = \frac{31\sqrt{3}}{9}F$$