ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{s^2+r^2-4rr_a}{p_a^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}=\frac{11}{2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$p_a=\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\to (1)$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 68; relation (•••); published at www.ssmrmh.ro)

$$\text{Now, } s^2-3r^2-16Rr\sin^2\frac{A}{2}=s^2+r^2-4r^2-\frac{16Rra(s-b)(s-c)}{4Rrs}$$

$$=s^2+r^2-\frac{4s(s-a)(s-b)(s-c)}{s^2}-\frac{4a(s-b)(s-c)}{s}$$

$$=s^2+r^2-\frac{4(s-b)(s-c)}{s}.(s-a+a)=s^2+r^2-4(s-b)(s-c)$$

$$=s^2+r^2-\frac{4(s-a)(s-b)(s-c)}{s-a}=s^2+r^2-\frac{4r^2s}{s-a}=s^2+r^2-4rr_a\overset{\text{via }(1)}{\Rightarrow}$$

$$p_a^2=\frac{4s^2}{(2s+a)^2}.(s^2+r^2-4rr_a)\Rightarrow\frac{s^2+r^2-4rr_a}{p_a^2}=\frac{(2s+a)^2}{4s^2}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{s^2+r^2-4rr_a}{p_a^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}=\sum_{cyc}\frac{(2s+a)^2}{4s^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}\overset{\text{Bogdan Fustei}}{=}$$

$$\frac{1}{4s^2}\left(12s^2+4s(2s)+2(s^2-4Rr-r^2)\right)+\sum_{cyc}\frac{r^2}{2r_ah_a}$$

$$=\frac{11s^2-4Rr-r^2}{2s^2}+\sum_{cyc}\frac{r^2}{\frac{r}{R}.s^2\sec^2\frac{A}{2}}=\frac{11s^2-4Rr-r^2}{2s^2}+\frac{Rr}{s^2}.\frac{4R+r}{2R}$$

$$=\frac{11s^2-4Rr-r^2+4Rr+r^2}{2s^2}\therefore\sum_{cyc}\frac{s^2+r^2-4rr_a}{p_a^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}=\frac{11}{2}$$

$\forall$ ABC (QED)