ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$m_a m_b m_c \geq \frac{(R+16r)s^2}{18}$$

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$$(m_a m_b m_c)^2 = \frac{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3}{16}$$

$$\left(\begin{array}{c}\text{Reference : Solution to Inequality in Triangle by}\\ \text{Dang Ngoc Minh} - 124;\text{ published at www.ssmrmh.ro}\end{array}\right) \overset{?}{\geq} \frac{(R+16r)^2 s^4}{324}$$

$$\Leftrightarrow 81s^6 - (4R^2 + 1100Rr - 1649r^2)s^4 - r^2(4860R^2 + 9720Rr + 2673r^2)s^2 -$$

$$81r^3(4R+r)^3 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and } \because 81(s^2 - 2R^2 - 8Rr - 3r^2)^3 \overset{Walker}{\geq} 0 \therefore \text{ in order}$$

$$\text{to prove } (*), \text{ it suffices to prove : LHS of } (*) \overset{?}{\geq} 81(s^2 - 2R^2 - 8Rr - 3r^2)^3$$

$$\Leftrightarrow (241R^2 + 422Rr + 1189r^2)s^4 -$$

$$(486R^4 + 3888R^3r + 11664R^2r^2 + 10692Rr^3 + 2430r^4)s^2 + 324R^6 + 3888R^5r +$$

$$17010R^4r^2 + 29808R^3r^3 + 23571R^2r^4 + 8262Rr^5 + 1053r^6 \underset{(**)}{\overset{?}{\geq}} 0 \text{ and } \because P =$$

$$(241R^2 + 422Rr + 1189r^2)(s^2 - 2R^2 - 8Rr - 3r^2)^2 \overset{Walker}{\geq} 0 \therefore \text{ in order}$$

$$\text{to prove } (**), \text{ it suffices to prove : LHS of } (**) \overset{?}{\geq} P$$

$$\Leftrightarrow (239R^4 + 828R^3r + 645R^2r^2 + 5432Rr^3 + 2352r^4)s^2 \underset{(***)}{\overset{?}{\geq}}$$

$$320R^6 + 2756R^5r + 9783R^4r^2 + 25940R^3r^3 + 44609R^2r^4 + 26304Rr^5 + 4824r^6$$

$$\text{and finally, LHS of } (***) \overset{Walker}{\geq}$$

$$(239R^4 + 828R^3r + 645R^2r^2 + 5432Rr^3 + 2352r^4)(2R^2 + 8Rr + 3r^2) \overset{?}{\geq}$$

$$RHS_{(***)} \Leftrightarrow 79t^6 + 406t^5 - 576t^4 - 3716t^3 + 2743t^2 + 4404t + 1116 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)^2(79t^4 + 722t^3 + 1996t^2 + 1380t + 279) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{Euler}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true and so, } m_a m_b m_c \geq \frac{(R+16r)s^2}{18}$$

$$\forall \text{ acute ABC, } " = " \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$