ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{AH}{r}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ge\frac{n_b}{h_b}+\frac{m_a}{h_c}+\frac{m_c}{h_a}+2$$

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$$n_a^2\overset{\text{Bogdan Fustei}}{=}s^2-2r_ah_a\therefore a^2n_a^2\overset{?}{\le}4(R-r)^2s^2$$

$$\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\le}4(R-r)^2s^2$$

$$\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\le}4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\ge}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\ge}0\to\text{true}\therefore n_a\le\frac{2(R-r)s}{a}=\frac{R-r}{r}.\frac{2rs}{a}\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1$$

$$\text{and analogs}\therefore\frac{n_b}{h_b}+\frac{m_a}{h_c}+\frac{m_c}{h_a}+2\le\frac{R}{r}+1+\frac{m_a}{h_c}+\frac{m_c}{h_a}\le\frac{2R}{r}+1$$

$\left(\because\frac{R}{r}\ge\frac{m_b}{h_c}+\frac{m_c}{h_b}\right.$ and analogs $\to$ reference : article titled "New Triangle Inequalities With Brocard's Angle" by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 − 7, published at : www.ssmrmh.ro$\left.\right)$

$$\overset{?}{\le}\frac{AH}{r}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\overset{\text{Bogdan Fustei}}{=}\frac{2R\cos A}{r}+\frac{n_a}{n_a-\frac{an_a}{s}}$$

$$(\because\Delta\text{ ABC is acute}\Rightarrow|2R\cos A|=2R\cos A)=\frac{2R\cos A}{r}+\frac{s-a+a}{s-a}$$

$$=\frac{2R\cos A}{r}+1+\frac{a}{s-a}\Leftrightarrow\frac{2R}{r}\overset{?}{\le}\frac{2R\cos A}{r}+\frac{a}{s-a}$$

$$\Leftrightarrow\frac{2R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left(2\sin^2\frac{A}{2}\right)\overset{?}{\le}\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\to\text{true}$$

$$\therefore\frac{AH}{r}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ge\frac{n_b}{h_b}+\frac{m_a}{h_c}+\frac{m_c}{h_a}+2\ \forall\ \Delta\text{ ABC},''=''\text{ iff }c=a\ (\text{QED})$$