ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{n_a^2+g_a^2}{rh_a}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\le 1+\frac{4R}{r}\left(\frac{r_b+r_c}{h_a}-1\right)$$

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$$r_b+r_c-h_a\overset{?}{\ge}\frac{b^2+c^2}{4R}\Leftrightarrow 4R\cos^2\frac{A}{2}\overset{?}{\ge}\frac{(b+c)^2}{4R}$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\overset{?}{\ge}4R\cos\frac{A}{2}\cos\frac{B-C}{2}\to\text{true}\because\cos\frac{B-C}{2}\le 1\therefore r_b+r_c-h_a\ge\frac{b^2+c^2}{4R}$$

$$\Rightarrow\frac{4R}{r}\left(\frac{r_b+r_c}{h_a}-1\right)\ge\frac{4R}{rh_a}.\frac{b^2+c^2}{4R}=\frac{b^2+c^2}{rh_a}$$

$$\Rightarrow\frac{n_a^2+g_a^2}{rh_a}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}-1-\frac{4R}{r}\left(\frac{r_b+r_c}{h_a}-1\right)$$

$$\overset{\text{Bogdan Fustei}}{=}\frac{(b-c)^2+2s(s-a)}{rh_a}+\frac{n_a}{n_a-\frac{an_a}{s}}-1-\frac{b^2+c^2}{rh_a}$$

$$=\frac{b^2+c^2}{rh_a}-\frac{2R.2bc}{r.bc}+\frac{2s(s-a)a}{2(s-a)(s-b)(s-c)}+\frac{s}{s-a}-1-\frac{b^2+c^2}{rh_a}$$

$$=-\frac{4R}{r}+\frac{sa}{(s-b)(s-c)}+\frac{a}{s-a}=-\frac{4R}{r}+\frac{a}{r^2s}.\left(s(s-a)+(s-b)(s-c)\right)$$

$$=-\frac{4R}{r}+\frac{a}{r^2s}.(s^2-sa-s^2+sa+bc)=-\frac{4R}{r}+\frac{4Rrs}{r^2s}=0$$

$$\therefore\frac{n_a^2+g_a^2}{rh_a}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\le 1+\frac{4R}{r}\left(\frac{r_b+r_c}{h_a}-1\right)\ \forall\Delta\text{ ABC},$$

$$''='' \text{ iff } b=c\ (\text{QED})$$