ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{(n_b+n_c)^2}{\sum_{cyc} h_a h_b} > \frac{R}{2r} \geq \frac{5R - r + \sum_{cyc}\frac{n_a g_a}{h_a}}{2\sum_{cyc} h_a}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{n_a g_a}{h_a} \overset{AM-GM}{\leq} \sum_{cyc}\frac{a(n_a^2+g_a^2)}{4rs} \overset{Bogdan\ Fustei}{=} \sum_{cyc}\frac{a\left(4m_a^2 - 2s(s-a)\right)}{4rs}$$

$$= \frac{1}{4rs}\cdot\left(\sum_{cyc}\left(a\left(2\sum_{cyc}a^2 - 3a^2\right)\right) - 2s\left(s(2s) - 2(s^2-4Rr-r^2)\right)\right)$$

$$= \frac{8s(s^2-4Rr-r^2) - 6s(s^2-6Rr-3r^2) - 4s(4Rr+r^2)}{4rs} = \frac{s^2-6Rr+3r^2}{2r}$$

$$\therefore 5R - r + \sum_{cyc}\frac{n_a g_a}{h_a} \leq 5R - r + \frac{s^2-6Rr+3r^2}{2r} = \frac{s^2+4Rr+r^2}{2r} = \frac{R}{r}\cdot\sum_{cyc} h_a$$

$$\Rightarrow \frac{R}{2r} \geq \frac{5R - r + \sum_{cyc}\frac{n_a g_a}{h_a}}{2\sum_{cyc} h_a} \text{ and again, } \frac{(n_b+n_c)^2}{\sum_{cyc} h_a h_b} > \frac{s^2}{\sum_{cyc} h_a h_b}$$

(Reference → "Nagel's Cevians Revisited (ii)" by Bogdan Fustei; published at : www.ssmrmh.ro)

$$= \frac{s^2}{\frac{4Rrs(2s)}{4R^2}} = \frac{R}{2r} \text{ and so, } \frac{(n_b+n_c)^2}{\sum_{cyc} h_a h_b} > \frac{R}{2r} \geq \frac{5R - r + \sum_{cyc}\frac{n_a g_a}{h_a}}{2\sum_{cyc} h_a}\ \forall\ \Delta\ ABC,$$

" = " iff Δ ABC is equilateral (QED)