ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$n_a \le AI + \sqrt{(b-c)^2 + r^2}$$

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$$n_a \overset{?}{\le} AI + \sqrt{(b-c)^2 + r^2}$$

$$\Leftrightarrow n_a^2 \boxed{\overset{?}{\underset{(*)}{\le}}} AI^2 + (b-c)^2 + r^2 + 2AI.\sqrt{(b-c)^2 + r^2}$$

Let s_0 = semiperimeter and then : $n_a^2 - AI^2 - (b-c)^2 - r^2 \overset{\text{Bogdan Fustei}}{=}$

$$s_0^2\left(1 - \frac{r}{R}\sec^2\frac{A}{2}\right) - 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2} -$$

$$16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} = 16R^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}\left(1 - \frac{2s(c-s)}{1-s^2}\right) -$$

$$4R^2(c-s)^2 - 16R^2s^2(1-c^2) - 4R^2s^2(c-s)^2 \left(s = \sin\frac{A}{2}, c = \cos\frac{B-C}{2}\right)$$

$$= 4R^2((s+c)^2(1+2s^2-2sc) - (c-s)^2 - 4s^2(1-c^2) - s^2(c-s)^2)$$

$$= 8R^2s(2(c-s) - c(c-s)(c+s))$$

$$\therefore n_a^2 - AI^2 - (b-c)^2 - r^2 \boxed{\overset{①}{=}} 8R^2s(c-s)(2-c^2-cs)$$

Also, $2AI.\sqrt{(b-c)^2 + r^2}$

$$= 8R\sin\frac{B}{2}\sin\frac{C}{2}.\sqrt{16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2} + 16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}$$

$$= 4R(c-s).\sqrt{16R^2s^2(1-c^2) + 4R^2s^2(c-s)^2}$$

$$\therefore 2AI.\sqrt{(b-c)^2 + r^2} \boxed{\overset{②}{=}} 8R^2s(c-s).\sqrt{(c-s)^2 + 4(1-c^2)}$$

$$\therefore ① \text{ and } ② \Rightarrow (*) \Leftrightarrow 2 - c^2 - cs \overset{?}{\le} \sqrt{(c-s)^2 + 4(1-c^2)}$$

$$\Leftrightarrow c^2 - c^4 + 2cs - 2sc^3 + s^2 - c^2s^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow c^2(1-c^2) + 2cs(1-c^2) + s^2(1-c^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow \left(1 - \cos^2\frac{B-C}{2}\right)\left(\sin\frac{A}{2} + \cos\frac{B-C}{2}\right)^2 \overset{?}{\ge} 0 \to \text{true} \because \sin\frac{A}{2}, \cos\frac{B-C}{2} > 0$$

$$\text{and } \cos^2\frac{B-C}{2} \le 1 \Rightarrow (*) \text{ is true} \therefore n_a \le AI + \sqrt{(b-c)^2 + r^2}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } b = c \text{ (QED)}$$