ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2\sum_{cyc}\frac{n_a}{\sqrt{4r^2+(n_a-g_a)^2}} \geq 3+\sum_{cyc}\frac{b+c}{a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(n_a-g_a)^2 = n_a^2+g_a^2-2n_ag_a \overset{\text{Bogdan Fustei}}{\leq}$$
$$(b-c)^2+2s(s-a)-2s(s-a) = (b-c)^2$$
$$\therefore \sqrt{4r^2+(n_a-g_a)^2} \leq \sqrt{4r^2+(b-c)^2} \overset{\text{Bogdan Fustei}}{=} \frac{an_a}{s} \Rightarrow \frac{2n_a}{\sqrt{4r^2+(n_a-g_a)^2}} \geq$$
$$\frac{2n_as}{an_a} = \frac{a+b+c}{a} \therefore \frac{2n_a}{\sqrt{4r^2+(n_a-g_a)^2}} \geq 1+\frac{b+c}{a} \text{ and analogs}$$
$$\therefore 2\sum_{cyc}\frac{n_a}{\sqrt{4r^2+(n_a-g_a)^2}} \geq 3+\sum_{cyc}\frac{b+c}{a}\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)