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In ΔABC the following relationship holds:

$$\sum\sqrt{2R\left(n_a-\sqrt{4r^2+(n_a-g_a)^2}\right)}\geq\sum AI\sqrt{\frac{n_a}{h_a}}$$

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Solution by Tapas Das-India

$$r=\frac{F}{s}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=(s-a)tan\left(\frac{A}{2}\right)\Rightarrow(s-a)=\frac{r}{\tan(\frac{A}{2})}\ (1)$$

$$lemma: |b-c|\geq n_a-g_a, \frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$$

(Reference: About Nagel and Gergonne's cevian by Bogdan Fustei , www.ssmrmh.ro)

$$L.H.S=\sum\sqrt{2R\left(n_a-\sqrt{4r^2+(n_a-g_a)^2}\right)}\overset{lemma}{\geq}$$

$$\geq\sum\sqrt{2R\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\overset{lemma}{\geq}$$

$$\geq\sum\sqrt{2R\left(n_a-\frac{n_a}{h_a}.2r\right)}=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R(h_a-2r)}=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R.\frac{2r(s-a)}{a}}=$$

$$=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R.\frac{2r(s-a)}{4Rsin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}}.\overset{(1)}{=}$$

$$=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R.\frac{2r\frac{r}{\tan(\frac{A}{2})}}{4Rsin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}}.=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{\frac{r^2}{\sin^2\left(\frac{A}{2}\right)}}.=\sum AI\sqrt{\frac{n_a}{h_a}}$$

Equality holds for an equilateral triangle.