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In any Δ ABC the following relationship holds :

$$\frac{n_a^2+g_a^2+r_ar-r_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}\le\frac{2Rr_a}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{n_a^2+g_a^2+r_ar-r_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}-\frac{2Rr_a}{r}\overset{\text{Bogdan Fustei}}{=}$$

$$\frac{r\left((b-c)^2+2s(s-a)+(s-b)(s-c)-s(s-a)\right)-2R.\frac{rs}{s-a}.\left(n_a-\frac{an_a}{s}\right)}{r\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}$$

$$=\frac{r\left((b-c)^2+s(s-a)+(s-b)(s-c)\right)-r.2Rn_a}{r\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\le$$

$$\frac{r\left((b-c)^2+s(s-a)+(s-b)(s-c)\right)-r.2R\left(\frac{b^2-bc+c^2}{2R}\right)}{r\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}$$

(Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba − 28; published at www.ssmrmh.ro)

$$=\frac{-bc+s(s-a)-s^2+sa+bc}{n_a-\sqrt{4r^2+(b-c)^2}}=0\ \therefore\ \frac{n_a^2+g_a^2+r_ar-r_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}\le\frac{2Rr_a}{r}$$

$\forall\,\Delta$ ABC," = " iff b = c (QED)