ROMANIAN MATHEMATICAL MAGAZINE

In any convex quadrilateral ABCD the following relationship holds

$$\sum_{cyc}\sqrt{\frac{a}{b+c+d-a}}\ge 2\sqrt{2}$$

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In any convex quadrilateral ABCD, $\sum_{cyc} a > 2a$ and analogs and so,

$$\sum_{cyc}\sqrt{\frac{a}{b+c+d-a}}=\sum_{cyc}\frac{a^2}{\sqrt{a\left(a^2\left(\sum_{cyc}a-2a\right)\right)}}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\sqrt{a\left(a^2\left(\sum_{cyc}a-2a\right)\right)}}\overset{\text{CBS}}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}\,.\sqrt{\sum_{cyc}\left(a^2\left(\sum_{cyc}a-2a\right)\right)}}\overset{?}{\ge}2\sqrt{2}$$

$$\Leftrightarrow\left(\sum_{cyc}a\right)^3\overset{?}{\ge}8\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)-16\sum_{cyc}a^3$$

$$\Leftrightarrow 9\sum_{cyc}a^3+6(abc+abd+acd+bcd)\overset{?}{\underset{(*)}{\ge}}$$
$$5(a^2b+ab^2+a^2c+ac^2+a^2d+ad^2+b^2c+bc^2+b^2d+bd^2+c^2d+cd^2)$$

Via Schur, $a^3+b^3+c^3+3abc\overset{①}{\ge}a^2b+b^2c+c^2a+ab^2+bc^2+ca^2$ and

$$a^3+b^3+d^3+3abd\overset{②}{\ge}a^2b+b^2d+d^2a+ab^2+bd^2+da^2,$$

$$a^3+c^3+d^3+3acd\overset{③}{\ge}ca^2+c^2d+d^2a+c^2a+cd^2+da^2 \text{ and}$$

$$b^3+c^3+d^3+3bcd\overset{④}{\ge}b^2c+c^2d+bd^2+bc^2+cd^2+b^2d\ \&\ ①+②+③+④$$

$$\Rightarrow 9\sum_{cyc}a^3+9(abc+abd+acd+bcd)\overset{(**)}{\ge}$$
$$6(a^2b+ab^2+a^2c+ac^2+a^2d+ad^2+b^2c+bc^2+b^2d+bd^2+c^2d+cd^2)$$

$(**)\Rightarrow$ in order to prove $(*)$, it suffices to prove :

$$a^2b+ab^2+a^2c+ac^2+a^2d+ad^2+b^2c+bc^2+b^2d+bd^2+c^2d+cd^2\overset{?}{\underset{(***)}{\ge}}$$

$3(abc+abd+acd+bcd)$ and now,

$(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)+(a^2b+b^2d+d^2a+ab^2+bd^2+da^2)+$
$(ca^2+c^2d+d^2a+c^2a+cd^2+da^2)+(b^2c+c^2d+bd^2+bc^2+cd^2+b^2d)$

$$\overset{AM-GM}{\geq} 6(abc + abd + acd + bcd)$$

$$\Rightarrow a^2b + ab^2 + a^2c + ac^2 + a^2d + ad^2 + b^2c + bc^2 + b^2d + bd^2 + c^2d + cd^2 \geq$$

$$3(abc + abd + acd + bcd) \Rightarrow (***) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \sqrt{\frac{a}{b+c+d-a}} \geq 2\sqrt{2}$$

$\forall$ convex quadrilateral ABCD, " = " iff ABCD is a rhombus (QED)