ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\arcsin\left(\frac{\sqrt{3}}{2}\right)+\arcsin\left(\frac{\sqrt{8}-\sqrt{3}}{6}\right)+\cdots+\arcsin\left(\frac{\sqrt{k^2+2k}-\sqrt{k^2-1}}{k(k+1)}\right)=cos^{-1}\left(\frac{1}{k+1}\right)$$

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Solution by Amin Hajiyev-Azerbaijan

$$S=\arcsin\left(\frac{\sqrt{3}}{2}\right)+\arcsin\left(\frac{\sqrt{8}-\sqrt{3}}{6}\right)+\cdots+\arcsin\left(\frac{\sqrt{k^2+2k}-\sqrt{k^2-1}}{k(k+1)}\right)$$

$$S=\sin^{-1}\left(\frac{\sqrt{2^2-1}-\sqrt{1^2-1}}{1(1+1)}\right)+\sin^{-1}\left(\frac{\sqrt{3^2-1}-\sqrt{2^2-1}}{2(2+1)}\right)+\cdots$$
$$+\sin^{-1}\left(\frac{\sqrt{(k+1)^2-1}-\sqrt{k^2-1}}{k(k+1)}\right)$$

$$S=\sum_{n=1}^{k}\sin^{-1}\left(\frac{\sqrt{(n+1)^2-1}-\sqrt{n^2-1}}{n(n+1)}\right)$$
$$=\sum_{n=1}^{k}\sin^{-1}\left(\sqrt{\frac{(n+1)^2-1}{n^2(n+1)^2}}-\sqrt{\frac{n^2-1}{n^2(n+1)^2}}\right)$$
$$=\sum_{n=1}^{k}\sin^{-1}\left(\frac{1}{n}\sqrt{1-\frac{1}{(n+1)^2}}-\frac{1}{n+1}\sqrt{1-\frac{1}{n^2}}\right)$$

Note: $\alpha=\arcsin(x)\,,\beta=\arcsin(y)\ \rightarrow\ \alpha,\beta\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$

$$cos^2(\alpha)=\sqrt{1-\sin^2(\alpha)}=\sqrt{1-x^2}$$

$$cos^2(\beta)=\sqrt{1-\sin^2(\beta)}=\sqrt{1-y^2}$$

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)=x\sqrt{1-y^2}-y\sqrt{1-x^2}$$

$$\alpha-\beta=\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$$

$$\arcsin(x)-\arcsin(y)=\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\ \rightarrow\ x=\frac{1}{n},\qquad y=\frac{1}{n+1}$$

$$S=\sum_{n=1}^{k}\left(\sin^{-1}\frac{1}{n}-\sin^{-1}\frac{1}{n+1}\right)$$

$$=\sin^{-1}1-\sin^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}-\sin^{-1}\frac{1}{3}+\cdots+\sin^{-1}\frac{1}{k}-\sin^{-1}\frac{1}{k+1}$$

$$S=\sin^{-1}1-\sin^{-1}\frac{1}{k+1}=\frac{\pi}{2}-\sin^{-1}\frac{1}{k+1}=cos^{-1}\left(\frac{1}{k+1}\right)$$