ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ *then prove that* :

$$\frac{(a+b)^3}{b+c}\sqrt{\frac{a+b}{b+c}}+\frac{(b+c)^3}{c+a}\sqrt{\frac{b+c}{c+a}}+\frac{(c+a)^3}{a+b}\sqrt{\frac{c+a}{a+b}}\ge 108\sqrt{3}\frac{(abc)^{\frac{7}{3}}}{(a^2+b^2+c^2)^{\frac{5}{2}}}$$

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$$\sum_{cyc}\left(\frac{(b+c)^3}{c+a}\sqrt{\frac{b+c}{c+a}.1}\right)\overset{GM-HM}{\ge}\sum_{cyc}\frac{2(b+c)^4}{(c+a)(b+c+c+a)}$$

$$=2\sum_{cyc}\frac{x^4}{y(x+y)}\ (x=b+c, y=c+a, z=a+b)\overset{Bergstrom}{\ge}\frac{2\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}xy+\sum_{cyc}x^2}\ge$$

$$\frac{2\left(\sum_{cyc}x^2\right)^2}{2\sum_{cyc}x^2}=\sum_{cyc}x^2=\sum_{cyc}(b+c)^2=2\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)\overset{AM-GM}{\ge}$$

$$2\left(3.\sqrt[3]{a^2b^2c^2}+3.\sqrt[3]{a^2b^2c^2}\right)=12(abc)^{\frac{2}{3}}\overset{?}{\ge}108\sqrt{3}.\frac{(abc)^{\frac{7}{3}}}{(a^2+b^2+c^2)^{\frac{5}{2}}}$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^{\frac{5}{2}}\overset{?}{\ge}9\sqrt{3}.(abc)^{\frac{5}{3}}\to\text{true}\because\left(\sum_{cyc}a^2\right)^{\frac{5}{2}}\overset{AM-GM}{\ge}\left(3(abc)^{\frac{2}{3}}\right)^{\frac{5}{2}}=9\sqrt{3}.(abc)^{\frac{5}{3}}$$

$$\text{and so, }\frac{(a+b)^3}{b+c}.\sqrt{\frac{a+b}{b+c}}+\frac{(b+c)^3}{c+a}.\sqrt{\frac{b+c}{c+a}}+\frac{(c+a)^3}{a+b}.\sqrt{\frac{c+a}{a+b}}$$

$$\ge 108\sqrt{3}.\frac{(abc)^{\frac{7}{3}}}{(a^2+b^2+c^2)^{\frac{5}{2}}}\ \forall a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$