ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x + y + z = xyz$ then prove that :

$$\sum_{cyc}\left(x(1+yz)\sqrt{1+x^2}\right) \ge 24\sqrt{3}$$

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$$\sum_{cyc}\left(x(1+yz).\sqrt{1+x^2}\right) = \sum_{cyc}\left((x+xyz).\sqrt{\frac{xyz}{x+y+z}+x^2}\right)$$

$$\left(\because 1 = \frac{xyz}{x+y+z}\right) = \sum_{cyc}\left((x+x+y+z).\sqrt{\frac{x}{x+y+z}.(yz+x^2+xy+xz)}\right)$$

$$(\because xyz = x+y+z) = \sum_{cyc}\left(((x+y)+(z+x)).\sqrt{\frac{x}{xyz}.(x+y)(z+x)}\right)$$

$$(\because x+y+z = xyz) \overset{AM-GM}{\ge} \sum_{cyc}\left(2.\sqrt{(x+y)(z+x)}.\frac{\sqrt{(x+y)(z+x)}}{\frac{y+z}{2}}\right)$$

$$= 4\sum_{cyc}\frac{(x+y)(z+x)}{y+z} \overset{AM-GM}{\ge} 12.\sqrt[3]{(x+y)(y+z)(z+x)} \overset{Cesaro}{\ge} 12.\sqrt[3]{8xyz} \overset{xyz\ge 3\sqrt{3}}{\ge}$$

$$12.\sqrt[3]{8(3\sqrt{3})}\left(\because xyz = \sum_{cyc} x \overset{AM-GM}{\ge} 3.\sqrt[3]{xyz} \Rightarrow \sqrt[3]{x^2y^2z^2} \ge 3 \Rightarrow xyz \ge 3\sqrt{3}\right)$$

$$= 24\sqrt{3} \therefore \sum_{cyc}\left(x(1+yz).\sqrt{1+x^2}\right) \ge 24\sqrt{3}\ \forall\, x,y,z > 0 \mid x+y+z = xyz,$$

$$'' ='' \text{ iff } x = y = z = \sqrt{3}\ (\text{QED})$$