

ROMANIAN MATHEMATICAL MAGAZINE

If $\rho^2 + \ln \rho = 0$, $a, b, c > \rho$ then:

$$\sqrt[3]{abc \cdot e^{a^2+b^2+c^2}} > 1 + \sqrt[3]{(a^2 + \ln a)(b^2 + \ln b)(c^2 + \ln c)}$$

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Let $f(x) = x^2 + \ln x$, $x > 0$ then $f'(x) = 2x + \frac{1}{x} > 0$ and $f(\rho) = 0$ (given)

so f is strictly increasing, as $a, b, c > \rho$
therefore $a^2 + \ln a > 0, b^2 + \ln b > 0, c^2 + \ln c > 0$

We need to show:

$$\sqrt[3]{abc \cdot e^{a^2+b^2+c^2}} > 1 + \sqrt[3]{(a^2 + \ln a)(b^2 + \ln b)(c^2 + \ln c)} \quad (1)$$

$$\text{Let } x = a^2 + \ln a, y = b^2 + \ln b, z = c^2 + \ln c$$

$$a^2 + b^2 + c^2 = x + y + z - \ln a - \ln b - \ln c = x + y + z - \ln abc$$

$$\text{so, } abc \cdot e^{a^2+b^2+c^2} = abc \cdot (e^{(x+y+z-\ln abc)}) = e^{x+y+z}$$

$$\text{hence inequality (1) becomes } \sqrt[3]{e^{x+y+z}} > 1 + \sqrt[3]{xyz}$$

$$\text{or } e^{\frac{x+y+z}{3}} > 1 + \sqrt[3]{xyz} \text{ or } e^{\sqrt[3]{xyz}} > 1 + \sqrt[3]{xyz} \text{ (AM - GM)}$$

$$\text{or } e^t - (1+t) > 0 \text{ or } 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots - (1+t) > 0$$

$$\frac{t^2}{2!} + \frac{t^3}{3!} + \dots > 0 \text{ True.}$$