ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\in(0,1]$ then:

$$\sqrt[3]{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}-1\right)}}+\sqrt[3]{3^{\left(b^{\frac{1}{b}}+\frac{1}{c}-1\right)}}+\sqrt[3]{3^{\left(c^{\frac{1}{c}}+\frac{1}{a}-1\right)}}\ge\sqrt[3]{9+8\sqrt[3]{3^{\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}+\frac{9}{a+b+c}\right)}}}$$

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Solution by Tapas Das-India

$$Let\ f(x)=x^{\frac{1}{x}}+\frac{1}{x},x\in(0,1]\ or\ f(u)=\left(\frac{1}{u}\right)^{u}+u=u^{-u}+u,$$

$$where\ u=\frac{1}{x}\ \&\ u\in[1,\infty)$$

$$f'(u)=1-\frac{1+\ln u}{u^u}=\frac{u^u-(1+\ln u)}{.u^u}\ ,clearly\ u^u>1\ as\ \ u\in[1,\infty)$$

$$Let\ h(u)=u^u-(1+\ln u)\ \text{then}\ h'(u)=u^u(1+\ln u)-\frac{1}{u}$$

$$u\ge1\ then\ u^u(1+\ln u)>1\ and\frac{1}{u}<1\ ,so\ h'(u)>1-1=0$$

$$so,h(u)increasing\ and\ h(u)>h(1)=0\ or,u^u-(1+\ln u)>0$$

$$so\ we\ can\ say\ f'(u)=\frac{u^u-(1+\ln u)}{.u^u}>0\ ,$$

$$so\ f(u)increasing\ \&\ f(u)\ge f(1),f(1)=2$$

$$or,f(u)\ge2\ \ or,u^{-u}+u\ge2\ or,x^{\frac{1}{x}}+x\ge2\ for$$

$$x\in(0,1]\ (1)\ ,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\overset{Bergstrom}{\ge}\frac{9}{a+b+c}$$

$$Let\ 3^{\left(a^{\frac{1}{a}}+\frac{1}{b}-1\right)}=\frac{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}\right)}}{3}=\frac{x}{3},\frac{3^{\left(b^{\frac{1}{b}}+\frac{1}{c}\right)}}{3}=\frac{y}{3},\frac{3^{\left(c^{\frac{1}{c}}+\frac{1}{a}\right)}}{3}=\frac{z}{3}$$

$$where\ x=3^{\left(a^{\frac{1}{a}}+\frac{1}{b}\right)},y=\ 3^{\left(b^{\frac{1}{b}}+\frac{1}{c}\right)},z=3^{\left(c^{\frac{1}{c}}+\frac{1}{a}\right)}$$

$$\frac{x+y+z}{3}\overset{AM-GM}{\ge}\sqrt[3]{xyz}=\sqrt[3]{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}\right)}3^{\left(b^{\frac{1}{b}}+\frac{1}{c}\right)}3^{\left(c^{\frac{1}{c}}+\frac{1}{a}\right)}}=\sqrt[3]{3^{\sum\left(a^{\frac{1}{a}}+\frac{1}{a}\right)}}\overset{(1)}{\ge}\sqrt[3]{3^6}=9\ (2)$$

$$R.H.S=\sqrt[3]{9+8\sqrt[3]{3^{\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}+\frac{9}{a+b+c}\right)}}}\le\sqrt[3]{9+8\sqrt[3]{3^{\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}}}=\sqrt[3]{9+8\sqrt[3]{xyz}}(A)$$

$$so\ L.H.S=\sqrt[3]{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}-1\right)}}+\sqrt[3]{3^{\left(b^{\frac{1}{b}}+\frac{1}{c}-1\right)}}+\sqrt[3]{3^{\left(c^{\frac{1}{c}}+\frac{1}{a}-1\right)}}=\sqrt[3]{\frac{x}{3}}+\sqrt[3]{\frac{y}{3}}+\sqrt[3]{\frac{z}{3}}$$

$$now,(L.H.S)^3=\frac{x+y+z}{3}+3\prod\left(\sqrt[3]{\frac{x}{3}}+\sqrt[3]{\frac{y}{3}}\right)\overset{Cesaro}{\ge}\frac{x+y+z}{3}+3\times8\sqrt[3]{\frac{xyz}{27}}$$

$$\overset{(2)}{\ge}9+8\sqrt[3]{xyz}\ \ so\ L.H.S\ge\sqrt[3]{9+8\sqrt[3]{xyz}}\ (B)$$

$$From\ (A)\ \&\ (B)we\ get\ L.H.S\ge R.H.S$$

Equality holds for a=b=c=1.