ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c,x,y,z>0$ and

$$2(a+b+c)^2+(x+y+z)^2+1\le 2(a+b+c)(x+y+z+1)$$

then:

$$\frac{(x+b+c)^9}{(x^ay^bz^c+a^xb^yc^z)^8}+\frac{(y+z+a)^9}{(x^by^cz^a+a^yb^zc^x)^8}+\frac{1}{(x^cy^az^b+a^zb^xc^y)^8}\ge\frac{19683}{256}$$

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$2(a+b+c)^2+(x+y+z)^2+1\le 2(a+b+c)(x+y+z+1)$ *can be written as*

$$\left(x+y+z-(a+b+c)\right)^2+\left((a+b+c)-1\right)^2\le 0$$

since L.H.S ≥ 0, so only possibility is

$$\left((x+y+z)-(a+b+c)\right)^2=0 \text{ and } \left((a+b+c)-1\right)^2=0$$

which implies that $x+y+z=a+b+c=1$ (1)

$$x^ay^bz^c+a^xb^yc^z+x^by^cz^a+a^yb^zc^x+x^cy^az^b+a^zb^xc^y=$$
$$=\left(x^ay^bz^c+a^xb^yc^z\right)+\left(x^by^cz^a+a^zb^xc^y\right)+\left(a^yb^zc^x+x^cy^az^b\right)$$

$$=\sum\left(x^ay^bz^c+a^xb^yc^z\right)\overset{AM-GM}{\le}\sum\left(\frac{ax+by+cz}{a+b+c}\right)^{a+b+c}+\sum\left(\frac{ax+by+cz}{x+y+z}\right)^{x+y+z}$$
$$\overset{(1)}{\le}\sum(ax+by+cz)+\sum(ax+by+cz)=2\sum(ax+by+cz)\ (2)$$

$$\frac{(x+b+c)^9}{(x^ay^bz^c+a^xb^yc^z)^8}+\frac{(y+z+a)^9}{(x^by^cz^a+a^yb^zc^x)^8}+\frac{1}{(x^cy^az^b+a^zb^xc^y)^8}=$$

$$=\frac{(x+b+c)^9}{(x^ay^bz^c+a^xb^yc^z)^8}+\frac{(y+z+a)^9}{(x^by^cz^a+a^yb^zc^x)^8}+\frac{1^9}{(x^cy^az^b+a^zb^xc^y)^8}\ge$$

$$\overset{Radon}{\ge}\frac{(x+b+c+y+z+a+1)^9}{(x^ay^bz^c+a^xb^yc^z+x^by^cz^a+a^yb^zc^x+x^cy^az^b+a^zb^xc^y)^8}\ge$$
$$\overset{(1)\&(2)}{\ge}\frac{(3)^9}{(2\sum(ax+by+cz))^8}=\frac{3^9}{\left(2(a+b+c)(x+y+z)\right)^8}\overset{(1)}{\ge}\frac{3^9}{2^8}=\frac{19683}{256}$$

Equality holds for $a=b=c=x=y=z=\frac{1}{3}$