ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0; abc = 1$ then:

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c \geq 8$$

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$$(a^3+1)^a(b^3+1)^b(c^3+1)^c \geq 8$$

$$\ln\left[(a^3+1)^a(b^3+1)^b(c^3+1)^c\right] \geq \ln 8$$

$$a\ln(a^3+1) + b\ln(b^3+1) + c\ln(c^3+1) \geq 3\ln 2 \quad \text{(to prove)}$$

Let be $f: (0,\infty) \to \mathbb{R}; f(x) = x\ln(1+x^3)$

$$f'(x) = \ln(1+x^3) + x \cdot \frac{3x^2}{1+x^3} = \ln(1+x^3) + \frac{3x^3}{1+x^3} > 0$$

$f'(x) > 0 \Rightarrow f$ increasing

$$f''(x) = \frac{3x^2}{1+x^3} + \frac{9x^2(1+x^3) - 3x^3 \cdot 3x^2}{(1+x^3)^2} = \frac{3x^2}{1+x^3} + \frac{9x^2}{(1+x^3)^2} > 0$$

$f''(x) > 0 \Rightarrow f$ convex

$$a\ln(a^3+1) + b\ln(b^3+1) + c\ln(c^3+1) =$$

$$= f(a) + f(b) + f(c) \overset{JENSEN}{\geq} 3f\left(\frac{a+b+c}{3}\right) \overset{AM-GM}{\geq}$$

$$\geq 3f\left(\sqrt[3]{abc}\right) = 3f\left(\sqrt[3]{1}\right) = 3f(1) = 3\ln 2$$

Equality holds for $a = b = c = 1$.