ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = 3$ then prove that:

$$4a^2b^2c^2 + 3 \geq \frac{21abc}{a+b+c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0$,
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then: $\sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} ab = 4Rr + r^2$ and then:

$$4a^2b^2c^2 + 3 \overset{?}{\geq} \frac{21abc}{a+b+c} \Leftrightarrow 4a^2b^2c^2 + \frac{3}{27}\left(\sum_{cyc} ab\right)^3 \overset{?}{\geq} \frac{21abc}{a+b+c}.\frac{1}{9}\left(\sum_{cyc} ab\right)^2$$

$$\left(\because \sum_{cyc} ab = 3\right) \Leftrightarrow s((4Rr + r^2)^3 + 36s^2r^4) \overset{?}{\geq} 21r^2 s(4Rr + r^2)^2$$

$$\Leftrightarrow 16R^3 - 72R^2r - 39Rr^2 - 5r^3 + 9rs^2 \underbrace{\overset{?}{\geq}}_{(*)} 0$$

Now, since $9rs^2 \overset{Gerretsen}{\geq} 9r(16Rr - 5r^2) \therefore$ LHS of $(*) \geq$

$$16R^3 - 72R^2r - 39Rr^2 - 5r^3 + 9r(16Rr - 5r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 16R^3 - 72R^2r + 105Rr^2 - 50r^3 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(4R - 5r)^2 \overset{?}{\geq} 0 \to \text{true}$$

$\because R \overset{Euler}{\geq} 2r \Rightarrow (*)$ is true $\forall \Delta XYZ_{s,R,r} \therefore 4a^2b^2c^2 + 3 \geq \frac{21abc}{a+b+c}$
$\forall a, b, c > 0 \,|\, ab + bc + ca = 3, '' ='' \text{ iff } a = b = c = 1$ (QED)