ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0, ab=1$ then:

$$a^{a^2+2b^2}\cdot b^{b^2+2a^2}\le 1$$

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$$a^{a^2+2b^2}\cdot b^{b^2+2a^2}\le 1\Rightarrow \ln\left(a^{a^2+2b^2}\cdot b^{b^2+2a^2}\right)\le \ln 1$$

$$(a^2+2b^2)\ln a+(b^2+2a^2)\ln b\le 0$$

$$\left(a^2+\frac{2}{a^2}\right)\ln a+\left(\frac{1}{a^2}+2a^2\right)\ln\frac{1}{a}\le 0$$

$$a^2\ln a+\frac{2}{a^2}\ln a+\frac{1}{a^2}\ln a^{-1}+2a^2\ln a^{-1}\le 0$$

$$a^2\ln a+\frac{2}{a^2}\ln a-\frac{1}{a^2}\ln a-2a^2\ln a\le 0$$

$$\left(a^2+\frac{2}{a^2}-\frac{1}{a^2}-2a^2\right)\ln a\le 0$$

$$\left(\frac{1}{a^2}-a^2\right)\ln a\le 0\Leftrightarrow \frac{a^4-1}{a^2}\ln a\ge 0$$

$$\frac{(a-1)(a+1)(a^2+1)}{a^2}\ln a\ge 0\Leftrightarrow (a-1)\ln a\ge 0$$

If $a\ge 1\Rightarrow a-1\ge 0; \ln a\ge 0\Rightarrow (a-1)\ln a\ge 0$

If $0<a\le 1\Rightarrow a-1\le 0; \ln a\le 0\Rightarrow (a-1)\ln a\ge 0$

Equality holds for $a=b=1$.