ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0, ab + bc + ca = 1$ then prove that :

$$\frac{a}{(a+1)^2} + \frac{b}{(b+1)^2} + \frac{c}{(c+1)^2} \geq \frac{1}{a+b+c} + \frac{abc}{4}$$

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When exactly one variable equals to zero WLOG we may assume $a = 0$ $(b, c > 0 \wedge bc = 1)$, then it boils down to proving : $\frac{b}{(b+1)^2} + \frac{c}{(c+1)^2} \underset{(*)}{\overset{?}{\geq}} \frac{1}{b+c}$

$$\Leftrightarrow \frac{bc(b+c) + 4bc + b + c}{(bc+b+c+1)^2} = \frac{2(b+c)+4}{(2+b+c)^2} = \frac{2}{b+c+2} \overset{?}{\geq} \frac{1}{b+c}$$

$\Leftrightarrow \boxed{b + c \overset{?}{\geq} 2 = 2\sqrt{bc}}$ $(\because bc = 1) \rightarrow$ true via AM $-$ GM and when : $a, b, c > 0$, then : $ab + bc + ca = 1 \Rightarrow 1 - bc = a(b+c) \overset{AM-GM}{\geq} 2a.\sqrt{bc}$

$$\Rightarrow \boxed{a \leq \frac{1-t^2}{2t}} \ (t = \sqrt{bc}) \therefore \sqrt{bc}(a+1) = ta + t \leq \frac{1-t^2}{2} + t \overset{?}{<} 2$$

$$\Leftrightarrow t^2 - 2t + 3 \overset{?}{>} 0 \Leftrightarrow (t-1)^2 + 2 \overset{?}{>} 0 \rightarrow \text{true} \therefore \sqrt{bc}(a+1) < 2 \Rightarrow \frac{1}{(a+1)^2} > \frac{bc}{4}$$

and so, $\boxed{\frac{a}{(a+1)^2} > \frac{abc}{4}}$ and $\frac{1}{a+b+c} \overset{a>0}{<} \frac{1}{b+c} \overset{\text{via } (*)}{\leq} \frac{b}{(b+1)^2} + \frac{c}{(c+1)^2}$

$$\Rightarrow \frac{a}{(a+1)^2} + \frac{b}{(b+1)^2} + \frac{c}{(c+1)^2} > \frac{1}{a+b+c} + \frac{abc}{4} \text{ and so,}$$

$$\frac{a}{(a+1)^2} + \frac{b}{(b+1)^2} + \frac{c}{(c+1)^2} \geq \frac{1}{a+b+c} + \frac{abc}{4} \ \forall\, a, b, c \geq 0 \,\big|\, ab + bc + ca = 1,$$

$$'' ='' \text{ iff } \begin{pmatrix} a = 0 \\ b = c = 1 \end{pmatrix} \text{ or } \begin{pmatrix} b = 0 \\ c = a = 1 \end{pmatrix} \text{ or } \begin{pmatrix} c = 0 \\ a = b = 1 \end{pmatrix} \text{ (QED)}$$