ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0; ab = a + b$ then:

$$\frac{a+1}{b^3} + \frac{b+1}{a^3} \geq \frac{3}{4}$$

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Denote: $S = a + b; P = ab; S > 0; P > 0$

$$a + b \overset{AM-GM}{\geq} 2\sqrt{ab} \Rightarrow S \geq 2\sqrt{P} \Rightarrow S \geq 2\sqrt{S} \Rightarrow$$

$$\Rightarrow S^2 \geq 4S \Rightarrow S \geq 4 \Rightarrow S - 4 \geq 0 \quad (1)$$

$$\frac{a+1}{b^3} + \frac{b+1}{a^3} \geq \frac{3}{4} \Leftrightarrow 4[a^3(a+1) + b^3(b+1)] \geq 3a^3b^3$$

$$4(a^4 + b^4) + 4(a^3 + b^3) \geq 3a^3b^3$$

$$4[(S^2 - 2P)^2 - 2P^2] + 4(S^3 - 3SP) - 3P^3 \geq 0$$

$$S = P \Rightarrow 4(S^2 - 2S)^2 - 8S^2 + 4S^3 - 12S^2 - 3S^3 \geq 0$$

$$4S^4 - 16S^3 + 16S^2 - 8S^2 + S^3 - 12S^2 \geq 0$$

$$4S^4 - 15S^3 - 4S^2 \geq 0 \Leftrightarrow 4S^2 - 15S - 4 \geq 0$$

$$4S^2 - 16S + S - 4 \geq 0 \Leftrightarrow 4S(S-4) + (S-4) \geq 0$$

$$(S-4)(4S+1) \geq 0 \Leftrightarrow S - 4 \geq 0 \text{ (True by (1))}.$$

Equality holds for $a = b = 2$.