ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ then prove that :

$$\sum_{cyc}\left((y+z).\sqrt{\frac{yz}{(x+y)(x+z)}}\right)\geq\sum_{cyc}x$$

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Assigning $y+z=a,z+x=b,x+y=c\Rightarrow a+b-c=2z>0$,
$b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c,b+c>a,c+a>b$
$\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius
$=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x=s$

$$\Rightarrow x=s-a,y=s-b,z=s-c\ \therefore\sum_{cyc}\left((y+z).\sqrt{\frac{yz}{(x+y)(x+z)}}\right)\overset{?}{\geq}\sum_{cyc}x$$

$$\Leftrightarrow\sum_{cyc}\left(a.\sqrt{\frac{(s-b)(s-c)}{bc}}\right)\overset{?}{\geq}s\Leftrightarrow\sum_{cyc}a\sin\frac{A}{2}\underbrace{\overset{?}{\geq}}_{(*)}s$$

$$\text{Now, }\sum_{cyc}a\sin\frac{A}{2}=\sum_{cyc}\left(\frac{a\sin\frac{A}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right)\geq\sum_{cyc}\left(a\sin\frac{A}{2}\cos\frac{B-C}{2}\right)$$

$$\left(\because 0<\cos\frac{B-C}{2}\leq 1\text{ and analogs}\right)=\frac{1}{2}\sum_{cyc}\left(a(\cos B+\cos C)\right)$$

$$=\frac{1}{2}\sum_{cyc}\left(a\left(\sum_{cyc}\cos A-\cos A\right)\right)=\frac{R+r}{2R}(2s)-\frac{R}{2}\sum_{cyc}\sin 2A$$

$$=\frac{s(R+r)}{R}-\frac{R}{2}.4.\frac{4Rrs}{8R^3}=s\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\left((y+z).\sqrt{\frac{yz}{(x+y)(x+z)}}\right)\geq\sum_{cyc}x\ \forall x,y,z>0,''='' \text{ iff } x=y=z\ (QED)$$