ROMANIAN MATHEMATICAL MAGAZINE

If $a^2b^2(1+a^3)(1+b^3)\ge 4$ then prove that :

$$a^2+b^2\ge 2$$

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$$\text{Since } xy\le\frac{(x+y)^2}{4}\ \forall\, x,y\in\mathbb{R}\ \therefore\ a^2b^2(1+a^3)(1+b^3)$$

$$=(a^2+a^2b^3)(b^2+b^2a^3)\le\frac{\left(a^2+b^2+a^2b^2(a+b)\right)^2}{4}$$

$$=\frac{(a^2+b^2)^2+a^4b^4(a+b)^2+2a^2b^2(a^2+b^2)(a+b)}{4}\le$$

$$\frac{(a^2+b^2)^2+a^4b^4.2(a^2+b^2)+2a^2b^2(a^2+b^2).\sqrt{2(a^2+b^2)}}{4}$$

$$(\because a^4b^4, a^2b^2(a^2+b^2)>0)\le$$

$$\frac{(a^2+b^2)^2+\frac{(a^2+b^2)^4}{16}.2(a^2+b^2)+\frac{(a^2+b^2)^2}{2}(a^2+b^2).\sqrt{2(a^2+b^2)}}{4}$$

$$(\because a^2+b^2>0)$$

$$\therefore 4\le a^2b^2(1+a^3)(1+b^3)\le\frac{\frac{m^4}{4}+\frac{1}{16}.\frac{m^8}{16}.m^2+\frac{1}{2}.\frac{m^4}{4}.\frac{m^2}{2}.m}{4}\ \left(m=\sqrt{2(a^2+b^2)}\right)$$

$$\Rightarrow m^{10}+16m^7+64m^4-4096\ge 0$$

$$\Rightarrow (m-2)\left(\begin{matrix}m^9+2m^8+4m^7+24m^6+48m^5+96m^4+256m^3+\\512m^2+1024m+2048\end{matrix}\right)\ge 0$$

$$\Rightarrow m=\sqrt{2(a^2+b^2)}\ge 2\ (\because m\ge 0)\Rightarrow a^2+b^2\ge 2$$

$$\text{whenever } a^2b^2(1+a^3)(1+b^3)\ge 4 \text{ and } a,b\in\mathbb{R},\text{''}=\text{'' iff } a=b=1\ (\text{QED})$$