ROMANIAN MATHEMATICAL MAGAZINE

If $a^3b^3(1+a^4)(1+b^4)\geq 4$, then prove that :

$$a^2+b^2\geq 2$$

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$$\text{Since } ab\leq\frac{a^2+b^2}{2}\ \forall\, a,b\in\mathbb{R} \text{ and } xy\leq\frac{(x+y)^2}{4}\ \forall\, x,y\in\mathbb{R}$$

$$\therefore a^3b^3(1+a^4)(1+b^4)=ab(b^2+b^2a^4)(a^2+a^2b^4)\leq$$

$$\frac{a^2+b^2}{2}.(b^2+b^2a^4)(a^2+a^2b^4)\ (\because b^2+b^2a^4, a^2+a^2b^4>0)$$

$$\leq\frac{a^2+b^2}{2}.\frac{\left(a^2+b^2+a^2b^2(a^2+b^2)\right)^2}{4}$$

$$=\frac{a^2+b^2}{2}.\frac{(a^2+b^2)^2+a^4b^4(a^2+b^2)^2+2a^2b^2(a^2+b^2)^2}{4}$$

$$\leq\frac{a^2+b^2}{2}.\frac{(a^2+b^2)^2+\frac{(a^2+b^2)^4}{16}.(a^2+b^2)^2+\frac{(a^2+b^2)^2}{2}(a^2+b^2)^2}{4}$$

$$(\because (a^2+b^2)^2>0)\ \therefore 4\leq a^3b^3(1+a^4)(1+b^4)\leq\frac{t\left(t^2+\frac{t^6}{16}+\frac{t^4}{2}\right)}{8}\ (t=a^2+b^2)$$

$$\Rightarrow t^7+8t^5+16t^3-512\geq 0$$

$$\Rightarrow (t-2)\left(t^6+2t^5+12t^4+24t^3+64t^2+128t+256\right)\geq 0$$

$$\Rightarrow t=a^2+b^2\geq 2\ (\because t>0)\ \therefore a^2+b^2\geq 2 \text{ whenever } a^3b^3(1+a^4)(1+b^4)\geq 4$$

$$\text{and } a,b\in\mathbb{R}, ''='' \text{ iff } (a=b=1) \text{ or } (a=b=-1)\ (\text{QED})$$