

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = 3$ then prove that :

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{75}{16a^2b^2c^2 + 9}$$

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Let $ab = x, bc = y, ca = z$ and then : $\sum_{cyc} \frac{1}{bc} \stackrel{?}{\geq} \frac{75}{16a^2b^2c^2 + 9}$

$$\Leftrightarrow \sum_{cyc} \frac{1}{x} \stackrel{?}{\geq} \frac{75}{16xyz + 9} \text{ subject to } \sum_{cyc} x = 3$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius = s, R, r

(say) and so, $2 \sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x \stackrel{(*)}{=} s \Rightarrow x = s - X, y = s - Y, z = s - Z$

$\therefore xyz \stackrel{(**)}{=} r^2s$ and hence $\sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y) \Rightarrow \sum_{cyc} xy \stackrel{(***)}{=} 4Rr + r^2$ and so,

$$(*) \stackrel{x+y+z=3}{\Leftrightarrow} \frac{\sum_{cyc} xy}{xyz} \stackrel{?}{\geq} \frac{\frac{75}{9} \cdot (\sum_{cyc} x)^2}{\frac{9}{27} \cdot (\sum_{cyc} x)^3 + 16xyz} = \frac{25(\sum_{cyc} x)^2}{(\sum_{cyc} x)^3 + 48xyz}$$

via $(*), (**)$ and $(***)$ $\frac{4Rr + r^2}{r^2s} \stackrel{?}{\geq} \frac{25s^2}{s^3 + 48r^2s} \Leftrightarrow (R - 6r)s^2 + 12r^2(4R + r) \stackrel{?}{\geq} 0$ (**)

and $(**)$ is trivially true if : $R - 6r \geq 0$ and so, we now focus on the case when :

$R - 6r < 0$ and then : LHS of $(**)$ $\stackrel{\text{Rouche}}{\geq}$

$$(R - 6r) \left(2R^2 + 10Rr - r^2 + 2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \right) + 12r^2(4R + r) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow (R - 2r)(2R^2 + 2Rr - 9r^2) \stackrel{?}{\geq} 2(6r - R)(R - 2r) \cdot \sqrt{R^2 - 2Rr} \text{ and } \therefore$$

$R - 2r \stackrel{\text{Euler}}{\geq} 0$ and $\therefore 2R^2 + 2Rr - 9r^2 = 2(R^2 - 4r^2) + 2r(R - 2r) + 3r^2 \stackrel{\text{Euler}}{\geq} 3r^2 > 0$ \therefore in order to prove $(***)$, it suffices to prove :

$$(2R^2 + 2Rr - 9r^2)^2 \stackrel{?}{>} 4(R^2 - 2Rr)(6r - R)^2$$

$$\Leftrightarrow 64t^3 - 272t^2 + 252t + 81 \stackrel{?}{>} 0 \left(t = \frac{R}{r} \right) \Leftrightarrow (4t + 1)(4t - 9)^2 \stackrel{?}{>} 0$$

\rightarrow true $\therefore t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**)$ is true $\forall \Delta XYZ_{s,R,r} \Rightarrow (*)$ is true

$$\therefore \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{75}{16a^2b^2c^2 + 9} \forall a, b, c > 0 \mid ab + bc + ca = 3,$$

" = " iff $a = b = c = 1$ (QED)