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If $a,b,c\in\mathbb{R}$ then:

$$\sum_{cyc}\sqrt{a^2+ab+b^2}\ge\sqrt{4\sum_{cyc}a^2+5\sum_{cyc}ab}$$

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$$\sqrt{a^2+ab+b^2}=\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}b}{2}\right)^2}$$

Via Minkowski's Inequality, we have:

$$\sum_{cyc}\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}b}{2}\right)^2}\overset{①}{\ge}\sqrt{\left(\sum_{cyc}\left(a+\frac{b}{2}\right)\right)^2+\left(\sum_{cyc}\frac{\sqrt{3}b}{2}\right)^2}$$

$$\text{Also,}\sum_{cyc}\left(a+\frac{b}{2}\right)=\frac{3}{2}(a+b+c)\qquad\text{and}\qquad\sum_{cyc}\frac{\sqrt{3}b}{2}=\frac{\sqrt{3}}{2}(a+b+c)$$

Substituting these back into ① :

$$LHS\ge\sqrt{\frac{9}{4}(a+b+c)^2+\frac{3}{4}(a+b+c)^2}=\sqrt{3(a+b+c)^2}=\sqrt{3\sum_{cyc}a^2+6\sum_{cyc}ab}$$

By the well − known inequality $\sum_{cyc}a^2\ge\sum_{cyc}ab$:

$$LHS\ge\sqrt{3\sum_{cyc}a^2+6\sum_{cyc}ab}\ge\sqrt{\left(4\sum_{cyc}a^2+5\sum_{cyc}ab\right)-\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}\ge$$

$$\ge\sqrt{4\sum_{cyc}a^2+5\sum_{cyc}ab}$$

Equality holds if and only if $a=b=c$.