ROMANIAN MATHEMATICAL MAGAZINE

**If $x, y, z > 0$ and $xyz = 1$ then prove that :**

$$\sum_{cyc} x^2 + 6 \geq \frac{3}{2}\left(\sum_{cyc} x + \sum_{cyc} \frac{1}{x}\right)$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

Let $\sqrt[2]{x} = a, \sqrt[3]{y} = b, \sqrt[3]{z} = c$ and then : $\sum_{cyc} x^2 + 6 \overset{?}{\geq} \frac{3}{2}\left(\sum_{cyc} x + \sum_{cyc} \frac{1}{x}\right)$

$\Leftrightarrow 2\left(\sum_{cyc} a^6 + 6a^2b^2c^2\right) \underset{(\bullet)}{\overset{?}{\geq}} 3\left(abc\sum_{cyc} a^3 + \sum_{cyc} a^3b^3\right)$ $(\because xyz = 1 \Rightarrow abc = 1)$ & assigning $b + c = X, c + a = Y, a + b = Z \Rightarrow X + Y - Z = 2c > 0, Y + Z - X = 2a > 0$ and $Z + X - Y = 2b > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle XYZ with semiperimeter, circumradius, inradius $= s, R, r$ (say);

and then : $a = s - X, b = s - Y, c = s - Z \therefore abc = r^2 s, \sum_{cyc} a = s,$

$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^3 = s^3 - 12Rrs$ & so, $\sum_{cyc} a^3b^3 = (4Rr + r^2)^3 - 12Rr^3s^2$

and $\sum_{cyc} a^6 = (s^3 - 12Rrs)^2 - 2((4Rr + r^2)^3 - 12Rr^3s^2)$ and hence, $(\bullet) \Leftrightarrow$

$2((s^3 - 12Rrs)^2 - 2((4Rr + r^2)^3 - 12Rr^3s^2) + 6r^4s^2) \overset{?}{\geq}$ $3(r^2s(s^3 - 12Rrs) + (4Rr + r^2)^3 - 12Rr^3s^2) \Leftrightarrow 2s^6 - (48Rr + 3r^2)s^4 +$ $r^2(288R^2 + 120Rr + 12r^2)s^2 - r^3(448R^3 + 336R^2r + 84Rr^2 + 7r^3) \underset{(*)}{\overset{?}{\geq}} 0$ and $\because$

$2(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*) \overset{?}{\geq} 2(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (48R - 33r)s^4 -$ $r(1248R^2 - 1080Rr + 138r^2)s^2 + r^2(7744R^3 - 8016R^2r + 2316Rr^2 - 257r^3)$ $\underset{(**)}{\overset{?}{\geq}} 0$ and $\because (48R - 33r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \overset{?}{\geq} (48R - 33r)(s^2 - 16Rr + 5r^2)^2$

$\Leftrightarrow (72R^2 - 144Rr + 48r^2)s^2 \underset{(***)}{\overset{?}{\geq}} r(1136R^3 - 2028R^2r + 1041Rr^2 - 142r^3)$

and finally, $\overbrace{(72R^2 - 144Rr + 48r^2)}^{= 72R(R-2r) + 48r^2 > 0} s^2 \overset{\text{Rouche}}{\geq}$

$$(72R^2-144Rr+48r^2)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)\overset{?}{\geq}$$
$$r(1136R^3-2028R^2r+1041Rr^2-142r^3)$$
$$\Leftrightarrow (R-2r)(144R^3-356R^2r+200Rr^2-47r^3)\overset{?}{\geq}$$
$$2(R-2r).\sqrt{R^2-2Rr}.(72R^2-144Rr+48r^2)$$
$$\Leftrightarrow (144R^3-356R^2r+200Rr^2-47r^3)^2\overset{?}{\geq}4(R^2-2Rr)(72R^2-144Rr+48r^2)$$
$$\left(\begin{array}{c}\because R-2r\overset{Euler}{\geq}0 \text{ and } \because 144R^3-356R^2r+200Rr^2-47r^3=\\ (R-2r)(34R(R-2r)+110R^2+64r^2)+81r^3\overset{Euler}{\geq}81r^3>0\end{array}\right)$$
$$\Leftrightarrow (4t+1)((2t-5)^2(148t(t-2)+140t^2+98)+156t-241)\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\rightarrow \text{true} \because t\overset{Euler}{\geq}2\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true }\forall\,\Delta\,XYZ_{s,R,r}\Rightarrow(\bullet)\text{ is true}$$
$$\therefore \sum_{cyc}x^2+6\geq\frac{3}{2}\left(\sum_{cyc}x+\sum_{cyc}\frac{1}{x}\right)\ \forall\,x,y,z>0\,\middle|\,xyz=1,$$
$$''='' \text{ iff } x=y=z=1\ (QED)$$